Q1 Test the convergence of the following integrals

(a) $I = \int_0^\infty e^{-x^2} x^{-3/4}\, dx$

Method 1

Let $x^2 = t$, then.

$2x\,dx = dt \Rightarrow dx = \frac{1}{2\sqrt{t}}\, dt$

$I = \int_0^\infty e^{-t}\, t^{-3/8}\, \frac{1}{2\sqrt{t}}\, dt$

(1 mark)

$= \frac{1}{2} \int_0^\infty e^{-t}\, t^{-7/8}\, dt$

$= \frac{1}{2} \int_0^\infty e^{-t}\, t^{1/8 - 1}\, dt$

$= \frac{1}{2} \Gamma\left(\frac{1}{8}\right)$

(1 mark)

Hence, convergent — (1 mark)

~~Metho~~

Method II

$$I = \underbrace{\int_0^1 e^{-x^2} x^{-3/4}\,dx}_{I_1} + \underbrace{\int_1^\infty e^{-x^2} x^{-3/4}\,dx}_{I_2}$$

(1 mark for breaking)

consider I_1, which has singularity at $x=0$

Let $g(x) = x^{-3/4}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} e^{-x^2} = 1 > 0$$

(1 mark to show I_1 cgt.)

and $\int_0^1 x^{-3/4}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-3/4}\,dx$

$= \lim_{\varepsilon\to 0} \left[4x^{1/4}\right]_\varepsilon^1$

$= 4$ (cgt)

So, I_1 is convergent by limit comparison test.

consider I_2,

Let ~~g(x) =~~ $h(x) = e^{-x^2}$

(1 mark to show I_2 cgt.)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{-3/4} = 0$$

$\left(\int_0^\infty e^{-x^2}\,dx \text{ is cgt.} \right)$

Hence, I_2 is convergent.

Q1 (b). $I = \int_0^\infty \frac{\sqrt{x}\, e^{-\sqrt{x}}}{\sin(x)}\, dx$

Singular points are $0,\ n\pi,\ n \in \mathbb{N}$

$$\left. I = \int_0^1 + \int_1^\pi + \int_\pi^\infty = I_1 + I_2 + I_3 \right\} \quad (1 \text{ mark})$$

Either

I₁ converges

$$f = \frac{\sqrt{x}\, e^{-\sqrt{x}}}{\sin(x)}, \quad g(x) = \frac{1}{\sqrt{x}}$$

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x}{\sin(x)} \cdot e^{-\sqrt{x}} = 1$$

and $\int_0^1 \frac{1}{\sqrt{x}} dx$ cgs

(2 marks to show I_1 cgt.)

OR

I_2 diverges

$$-f(x) = -\frac{\sqrt{x}}{\sin(x)} e^{-\sqrt{x}} \qquad g = \frac{1}{x - \pi}$$

$$\lim_{x \to \pi^-} \frac{-f(x)}{g(x)} = \lim_{x \to \pi^-} -\frac{\sqrt{x}\, e^{-\sqrt{x}}}{\sin(x)}(x - \pi)$$

$$= \sqrt{\pi}\, e^{-\sqrt{\pi}}$$

and $\int_1^\pi g(x)\, dx$ diverges

$\Rightarrow I_2$ diverges

(2 marks to show I_2 dgt.)

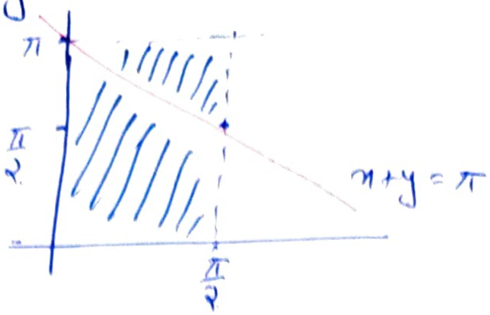
$\therefore$ From the defⁿ of improper integral I diverges

Method 1!

Ques 2

a)

$$I = \int_0^{\pi}\int_0^{\frac{\pi}{2}} |\sin(x+y)|\, dx\, dy$$



$$I = \int_0^{\frac{\pi}{2}}\int_0^{\pi - x} \sin(x+y)\, dy\, dx + \int_0^{\frac{\pi}{2}}\int_{\pi-x}^{\pi} (-\sin(x+y))\, dy\, dx \quad \text{———} \quad \boxed{3 \text{ marks}}$$

$$= \int_0^{\frac{\pi}{2}} \Big[-\cos(x+y)\Big]_0^{\pi-x} dx + \int_0^{\frac{\pi}{2}} \Big[\cos(x+y)\Big]_{\pi-x}^{\pi} dx$$

$$= -\int_0^{\frac{\pi}{2}} \big[(-1) - \cos x\big]\, dx + \int_0^{\frac{\pi}{2}} \big[-\cos x - (-1)\big]\, dx \quad \boxed{1 \text{ marks}}$$

$$= \int_0^{\frac{\pi}{2}} dx + \int_0^{\frac{\pi}{2}} \cancel{\cos x}\, dx - \int_0^{\frac{\pi}{2}} \cancel{\cos x}\, dx + \int_0^{\frac{\pi}{2}} dx$$

$$= 2\int_0^{\frac{\pi}{2}} dx = \pi.$$

Method-2

Let $u = x + y$, $v = y$ $\Rightarrow J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$ $\boxed{1 \text{ marks}}$

$\because 0 \le x \le \frac{\pi}{2}$, $0 \le y \le \pi$

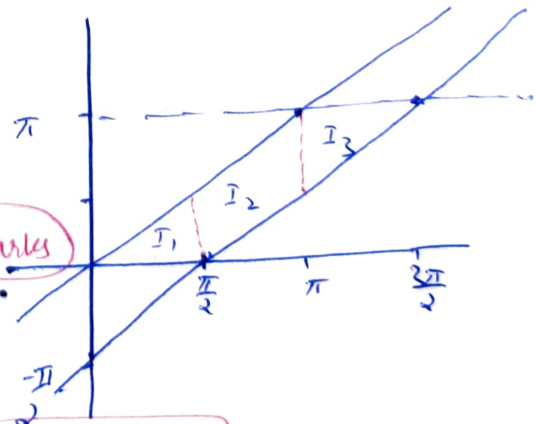
$\Rightarrow v \le u \le v + \frac{\pi}{2}$ & $0 \le v \le \pi$

$I = I_1 + I_2 + I_3$

where

$$I_1 = \int_0^{\frac{\pi}{2}}\int_0^{u} \sin u\, dv\, du = 1 \quad \boxed{1 \text{ marks}}$$

$$I_2 = \int_{\frac{\pi}{2}}^{\pi}\int_{u-\frac{\pi}{2}}^{u} \sin u\, dv\, du = \frac{\pi}{2} \quad \boxed{1 \text{ marks}}$$



$$I_3 = \int_{\pi}^{\frac{3\pi}{2}} \int_{u-\frac{\pi}{2}}^{\pi} (-\sin u)\, dv\, du = \frac{\pi}{2} - 1 \quad \boxed{1 \text{ marks}}$$

$$\therefore \; I = 1 + \frac{\pi}{2} + \frac{\pi}{2} - 1 = \pi.$$

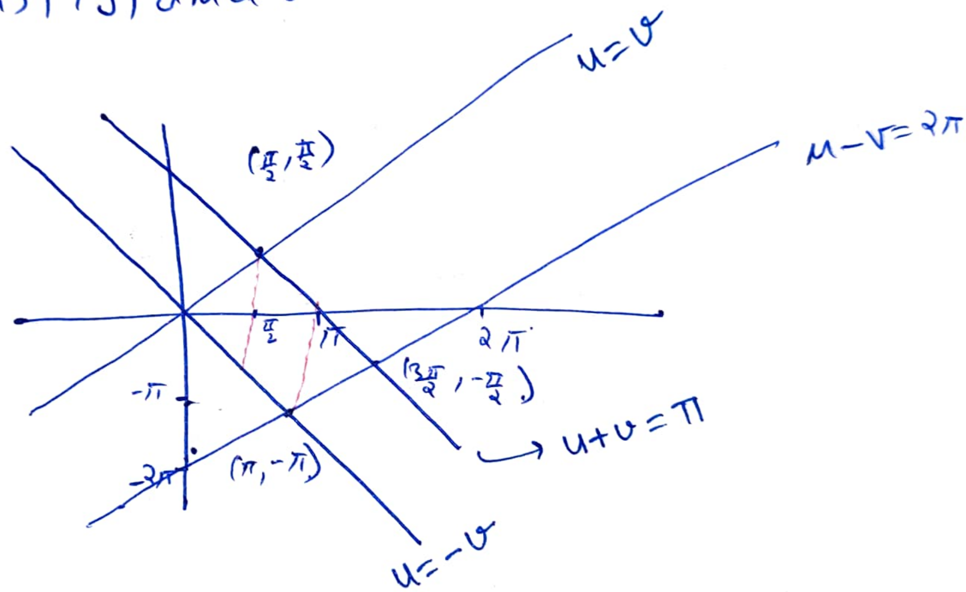
Method-3

Let $\left.\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}$

$$\Rightarrow J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \quad \boxed{1 \text{ marks}}$$

$$\Rightarrow I = \iint_D |\sin(u)|\,|J|\, du\, dv$$

where D :



$$I = I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^{\frac{\pi}{2}} \int_{-u}^{u} \frac{\sin(u)}{2}\, du\, dv = 1 \quad \boxed{1 \text{ marks}}$$

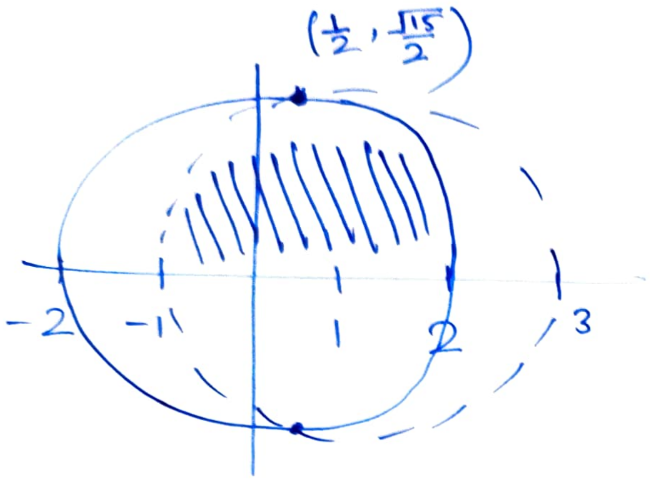
$$I_2 = \int_{\frac{\pi}{2}}^{\pi} \int_{-u}^{\pi-u} \frac{\sin(u)}{2}\, du\, dv = \frac{\pi}{2} \quad \boxed{1 \text{ marks}}$$

$$I_3 = \int_{\pi}^{\frac{3\pi}{2}} \int_{u-2\pi}^{\pi-u} \frac{(-\sin u)}{2}\, du\, dv = \frac{\pi}{2} - 1 \quad \boxed{1 \text{ marks}}$$

$$\therefore \; I = 1 + \frac{\pi}{2} + \frac{\pi}{2} - 1 = \pi.$$

Ques 2 b) $\iint_D xy \, dx\,dy$

where $D = \{(x, y) ;\ y \geq 0 ,\ x^2+y^2 \leq 4,\ (x-1)^2+y^2 \leq 4\}$



Point of intersection:-

$$x^2+y^2 = (x-1)^2+y^2$$

$$\Rightarrow \quad 2x = 1 \quad \Rightarrow \boxed{x = 1/2}$$

and $$y^2 = 4 - \frac{1}{4} = \frac{15}{4}$$

$$\Rightarrow \boxed{y = \frac{\sqrt{15}}{2}}$$

$\boxed{1 \text{ marks}}$

* Method I :-

$$\int_0^{\frac{\sqrt{15}}{2}} \int_{1-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy \, dx\,dy$$ —— $\boxed{1 \text{ marks}}$

$$= \int_0^{\frac{\sqrt{15}}{2}} y \left. \frac{x^2}{2} \right|_{1-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \frac{1}{2} \int_0^{\frac{\sqrt{15}}{2}} y \left[2\sqrt{4-y^2} - 1 \right] dy$$

Take $4-y^2 = t$

$-2y\,dy = dt$

$$\Rightarrow \frac{1}{4}\int_{1/4}^{4} 2\sqrt{t}-1\,dt = \frac{27}{16}$$ $\boxed{1 \text{ marks}}$

* Method II :-

$$\int_{-1}^{1/2}\int_{0}^{\sqrt{4-(x-1)^2}} xy\,dx\,dy + \int_{1/2}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dx\,dy$$

$\boxed{1 \text{ marks}}$

↳ Similarly calculating this correctly gives $\boxed{1 \text{ marks}}$

Q 3.a) The given cone is $z = \sqrt{x^2+y^2}$

$\therefore$ Let the function be,

$$f : z - \sqrt{x^2+y^2}$$

We know the surface area is given by,

$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} \, dA$$

where, $\hat{p}$ is the unit normal vector to the projected surface

1

Let us take the projection to the xy-plane

$\therefore \hat{p} = \hat{k}$ & $\nabla f = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right)$

$$\therefore S = \int_{x=0}^{1} \int_{y=x^2}^{x} \frac{\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}}{1} \, dy\,dx$$

1

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \sqrt{2} \, dy\,dx$$

$\int^{1}$

$$= \sqrt{2} \Big/ \left(x - x^2\right)$$

$$x=0$$

$$= \sqrt{2}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{3\sqrt{2}}$$

#

<u>Alternative Approach</u>

Taking $x = r\cos\theta$ & $y = r\sin\theta$

$y = x \Rightarrow r\sin\theta = r\cos\theta$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

& $y = x^2 \Rightarrow r\sin\theta = r^2\cos^2\theta$

$\Rightarrow r = \frac{\sin\theta}{\cos^2\theta}$

Now, $T(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\hat{k}$

& so, $T_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + \hat{k}$

$T_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$

$\therefore |T_r \times T_\theta| = r\sqrt{2}$

1

1

The surface area is,

$$S = \iint_R |T_r \times T_\theta| \, dr \, d\theta \quad 1$$

$$= \int_{\theta=0}^{\pi/4} \int_{r=0}^{\frac{\sin\theta}{\cos^2\theta}} r\sqrt{2} \, dr \, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{\sqrt{2} r^2}{2} \right]_0^{\frac{\sin\theta}{\cos^2\theta}} d\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \, d\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec^2\theta \tan^2\theta \, d\theta \quad 1$$

Let, $\tan\theta = x$

$\Rightarrow \sec^2\theta \, d\theta = dx$

$$\therefore S = \frac{1}{\sqrt{2}} \int_0^1 x^2 \, dx$$

$$= \frac{1}{\sqrt{2}} \left. \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3\sqrt{2}} \quad \#$$

Q) 3.b)

f is continuous on closed and bounded interval

$\therefore f$ is uniformly continuous.

$\therefore \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x-y| < \delta$. $\quad$ $|(b-a)$ is constant.

(A)

1

Now, a function f is Riemann integrable if for a given $\varepsilon > 0$, $\exists$ a partition P_ε s.t.

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

1

Now, let $P_\varepsilon = \{a = x_0, \ldots, x_n = b\}$ s.t.

$$\max_{0 \le k \le n} |x_k - x_{k-1}| < \delta$$

(AA)

1

$$\therefore \left| U(P_\varepsilon, f) - L(P_\varepsilon, f) \right| = \left| \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \right|$$

where, M_i is supremum of f in $[x_{i-1}, x_i]$

& m_i is infimum of f in $[x_{i-1}, x_i]$

& as f is U.C., M_i & m_i are attained in the intervals ie., $\exists\, x_i'$ & $x_i'' \in [x_{i-1}, x_i]$

s.t. $\quad M_i = f(x_i')$ & $m_i = f(x_i'')$

$$\therefore \left| U(P_\varepsilon, f) - L(P_\varepsilon, f) \right| = \left| \sum_{i=1}^{n} \left(f(x_i') - f(x_i'') \right) \Delta x_i \right|$$

$$\leq \sum_{i=1}^{n} \left| f(x_i') - f(x_i'') \right| \left| \Delta x_i \right|$$

$$\leq \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \left| x_i - x_{i-1} \right| \quad \text{by [illegible]}$$

$$= \frac{\varepsilon}{b-a} \left| b-a \right|$$

$$= \varepsilon$$

$\therefore f$ is Riemann integrable. #

1

4) Using Lagrange multiplier method, find the distance from origin to the curve of intersection of surfaces $x-y=1$ and $y^2-z^2=1$.

<u>Solution</u> Distance of $(x,y,z) \in \mathbb{R}^3$ to origin is $\sqrt{x^2+y^2+z^2}$. So we need to find out the minimum of $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints

$g_1(x,y,z) = x-y-1$

and $g_2(x,y,z) = y^2-z^2-1$.

The Lagrange multiplier equation is $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ where

$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g_1 = (1,-1,0)$$
$$\nabla g_2 = (0, 2y, -2z)$$

So, $(2x, 2y, 2z) = (\lambda_1, -\lambda_1 + 2y\lambda_2, -2z\lambda_2)$. (1)

Hence we have,

$$2x = \lambda_1,$$
$$2y = -\lambda_1 + 2y\lambda_2,$$
$$2z = -2z\lambda_2.$$

Also, $x-y=1$ and $y^2-z^2=1$.

If $z \neq 0$, then $2z(1+\lambda_2)=0$ implies $\lambda_2=-1$.

So $\lambda_1=-4y$.

Also $\lambda_1=2x$.

So $2x=-4y$. i.e. $x=-2y$.

Now $x-y=1$. So, $y=-\frac{1}{3}$.

So $x=\frac{2}{3}$.

As $y^2-z^2=1$

we get $z^2=\frac{1}{9}-1=-\frac{8}{9}$, not possible.

Hence $z=0$.

1

Now $z=0$ implies $y^2=1$.

So $y=\pm 1$.

$y=1 \Rightarrow x=2$.

$y=-1 \Rightarrow x=0$.

1

Hence we have two extremum points

$(2,1,0)$, $(0,-1,0)$.

$f(2,1,0)=5$

$f(0,-1,0)=1$

So the desired distance is 1.

1

Alternate solution

we need to minimize $f(x,y,z)=x^2+y^2+z^2$ subject to $x-y=1$ and $y^2-z^2=1$.

$$g(x,y,z) = x-y-y^2+z^2.$$

$$\nabla f = \lambda \nabla g$$

$$(2x, 2y, 2z) = (\lambda, -\lambda-2y\lambda, 2\lambda z)$$

(1)

So $2x=\lambda$

$2y=-\lambda-2y\lambda$ i.e. $2y(1+\lambda)=-\lambda$

$2z=2z\lambda$ i.e. $(\lambda-1)z=0$

so $z=0$ or $\lambda=1$.

If $z=0$, then $y=\pm 1$.

$y=1 \Rightarrow x=2$.

$y=-1 \Rightarrow x=0$.

(1)

If $\lambda=1$, then $x=\frac{1}{2}$, $y=-\frac{1}{2}$.

so $z^2=\frac{1}{4}-1<0$, not possible. (1)

so $(2,1,0)$, $(0,-1,0)$ are the extremum points and $f(2,1,0)=5$, $f(0,-1,0)=1$.

Hence the distance is 1. (1)

Q5(a) Show that a bounded increasing sequence is convergent.

Let $\{a_n\}$ be a bounded increasing sequence.

Since $\{a_n : n \in \mathbb{N}\}$ is bounded above, $\sup\limits_{n \in \mathbb{N}} a_n$ exists.

We'll show that $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\in\mathbb{N}} a_n$ [1 mark]

Let $M = \sup\limits_{n\in\mathbb{N}} a_n$ and let $\varepsilon > 0$.

Since $M-\varepsilon$ is not an upper bound of $\{a_n\}$,

$\exists\, n_0 \in \mathbb{N}$ s.t. $a_{n_0} > M-\varepsilon$

Since $\{a_n\}$ is increasing, $a_n \geq a_{n_0} > M-\varepsilon \;\; \forall n \geq n_0$ [1 mark]

Also, $a_n \leq M < M+\varepsilon \quad \forall n$

$\therefore \quad M-\varepsilon < a_n < M+\varepsilon \quad \forall n \geq n_0$ [1 mark]

$\Rightarrow \quad \lim\limits_{n\to\infty} a_n = M.$

Q5(b) $$\frac{\cos^2 n}{\sqrt{n}} = \frac{1}{2\sqrt{n}} + \frac{\cos(2n)}{2\sqrt{n}} \quad \text{—— (i)}$$

Taking $a_n = \cos(2n)$, $b_n = \frac{1}{2\sqrt{n}}$, we see that $b_n > 0 \;\forall n$, $\{b_n\}$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$

Also, $\left\{\sum\limits_{k=1}^{n} a_k\right\}_{n\in\mathbb{N}} = \left\{\sum\limits_{k=1}^{n} \cos(2k)\right\}_{n\in\mathbb{N}}$ is bounded. [1 mark]

$\therefore$ By the Dirichlet's test $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{\cos(2n)}{2\sqrt{n}}$ converges [1 mark]

Also, $\sum\limits_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges. [1 mark]

$\therefore$ From (i), $\sum\limits_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n}}$ diverges.

Some common mistakes for which no marks can be given:

(1) Cauchy condensation test is not applicable as $\left\{\frac{\cos^2 n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is not a decreasing sequence.

(2) Root test / ratio test fails.

(3) Limit comparison test does not work with say $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ or $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ as $\lim\limits_{n\to\infty} \cos^2 n$ or $\lim\limits_{n\to\infty} \sqrt{n}\cos^2 n$ does not exist.

(4) $\inf\{\cos^2 n\} = 0$, so comparison test with $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ fails.

(5) Just writing the series is divergent with incorrect reasoning gets zero marks.

(a) $f(x) = x^2(1-x) + \sin x$
$= x^2 - x^3 + \sin x$
$f(0) = 0$

Now $f'(x) = 2x - 3x^2 + \cos x$
$f'(0) = 1$

$f''(x) = 2 - 6x - \sin x$

So, approximation of $f(x)$ by its linear Taylor polynomial around 0 is

$$f(x) \simeq f(0) + x f'(0)$$

$$f(x) \simeq x$$

Error estimate:-

$$|R_2(x)| = \left| \frac{1}{2!} f''(c)\, x^2 \right| ;$$

where $c \in (0, x)$

$$= \left| \frac{1}{2!} (2 - 6c - \sin c)(x^2) \right|$$

$$\leq \frac{1}{2} \left(|2| + |6c| + |\sin c| \right) |x^2|$$

$$\leq \frac{1}{2} (2 + 6 + 1) = \frac{9}{2} .$$

(b) Let $f(x) = \sqrt{1+x}$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{1+x}}$$

Apply LMVT on interval $(0, x)$

$\exists \quad c \in (0, x)$ s.t.

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

$$\frac{\sqrt{1+x}-1}{x-0} = \frac{1}{2\sqrt{1+c}}$$

Since $c \in (0, x) \Rightarrow \dfrac{1}{\sqrt{1+c}} < 1$

$$\Rightarrow \frac{\sqrt{1+x}-1}{x} = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

$$\Rightarrow \sqrt{1+x} < 1+\frac{1}{x}$$

Alternate Solution $\Rightarrow$

$$\left(1+\frac{x}{2}\right)^2 = 1+x+\frac{x^2}{4} > 1+x$$

$$\left(\because x>0 \Rightarrow \frac{x^2}{4}>0\right)$$

Taking Square Root on Both Sides

$$1+\frac{x}{2} > \sqrt{1+x}.$$