Chapter 3
Elliptic Equations

3.1 Green's Identities

In this section we recall the Divergence theorem and note some consequences which are important for stating the Dirichlet and Neumann problems: Let us consider an open domain Ω and a vector field F defined in the closure Ω and differentiable in Ω,

$$\int_{\Omega}(\nabla\cdot F)dx=\int_{\partial\Omega}(F\cdot n)dS$$

where $\partial\Omega$ stands for the boundary of Ω and n is the unit outward normal vector to this boundary, and the integral on the right hand side is the surface integral. Taking $F=v\nabla u$ and using the identity

$$\nabla\cdot(v\nabla u)=\nabla u\cdot\nabla v+v\Delta u.$$

and the above divergence theorem, we get

$$\int_{\partial\Omega}v\frac{\partial u}{\partial n}dS=\int_{\Omega}\nabla u\cdot\nabla v dx+\int_{\Omega}v\Delta u dx$$

Now taking $\nabla u=(0,...w,0,...,0)$ with w at i^th place, we get $\Delta u=\frac{\partial w}{\partial x_i}$ and **the integration by parts formula:**

$$\int_{\Omega}v\frac{\partial w}{\partial x_i}dx=-\int_{\Omega}w\frac{\partial v}{\partial x_i}dx+\int_{\partial\Omega}vwn_i dS \tag{1.1}$$

where n_i is the i^th component of the unit outward normal. From these identities we can also obtain the Green's identity

$$\int_{\partial\Omega}(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n})dS=\int_{\Omega}(v\Delta u-u\Delta v)dx. \tag{1.2}$$

If we choose $v=1$ we get an important identity

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}dS=\int_{\Omega}\Delta u dx \tag{1.3}$$

If we take $u=v$ we get

$$\int_{\partial\Omega}u\frac{\partial u}{\partial n}dS=\int_{\Omega}(u\Delta u+|\nabla u|^2)dx. \tag{1.4}$$

Theorem 3.1.1 Uniqueness: *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta u = 0$ in Ω and either $u = 0$ or $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then from* (1.4) *we get that u is constant in $\overline{\Omega}$.*

Also let us recall the equation of continuity from the flow through a uniform medium. Suppose $u(x,t)$ measures the temperature at the point x and at the point t, then we have the equation

$$\frac{\partial}{\partial t}(uA) + \frac{\partial}{\partial x}(qA) = f(x,t)$$

where $q(x,t)$ is the flux. In case of heat conduction it observed that the flux follows the Darcy's law which says that flux is proporational to $\frac{\partial u}{\partial x}$

$$q(x,t) = -\beta\frac{\partial u}{\partial x},\ \beta > 0$$

Hence we obtain the heat conduction equation

$$\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial^2 x} = f(x,t)$$

where $f(x,t)$ represent the internal sources. There wil be infinitely many functions satisfying the above equation. But the temparature distribution is unique. So we look for boundary and initial conditions from the physical model. The initial tempartur distribution is $u(x,0)$ which can be measured. For the boundary conditions:

Forced boundary conditions: If we assume that the temparature at the both end points $x = a$ and $x = b$ is maintained at fixed. Say $u(a,t) = T_1$ and $u(b,t) = T_2$.

Natural boundary conditions: Suppose we dont maintain any fixed temparature, then the natural diffusive property of heat implies the flux at these ends are known. That is $\frac{\partial u}{\partial x}(a,t) = T_0(t)$ and $\frac{\partial u}{\partial x}(a,t) = T_1(t)$.

Steady state temparature: This the state where the tempature does not change with time. That is when $\frac{\partial u}{\partial t} = 0$. In this case the steady state temparature for forced boundary conditions satisfies the following:

Dirichlet Problem: Given the functions f and g, find the u that satisfies

$$\begin{aligned} -\Delta u(x) &= f(x), \text{ in } \Omega \\ u(x) &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

In case of natural boundary conditions, the steady state tempartures satisfy the problem problem:

Neumann Problem: Given the functions f and g, find the u that satisfies

$$\begin{aligned} -\Delta u(x) &= f(x), \text{ in } \Omega \\ \frac{\partial u}{\partial n}(x) &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative of u on the boundary.

The following equivalent formulation of optimization problem. In case of $\mathbb{R}^n$ it is easy to show such optimization problems has solution using the Lagrange-multiplier methods. In case of PDEs the problems is posed on infinite dimensional space. These formulations are often used for computing solutions using numerical methods known as Galarkin methods.

Dirichlet Principle:

Consider the Dirichlet problem

$$-\Delta u = f(x) \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega \tag{1.5}$$

and the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx$$

over $\mathscr{A} = \{u \in C^2(\Omega) \cap C(\overline{\Omega}), u = 0 \ on \ \partial\Omega\}$. We consider the minimization problem

$$I(u) = \min_{w \in \mathscr{A}} I(w) \tag{1.6}$$

Then we have the following

Theorem 3.1.2 *The problem* (1.5) *has a solution in* $\mathscr{A}$ *if and only if* u *solves the problem in* (1.6).

Proof. $\Longrightarrow$: If u solves the problem in (1.5). Then for any $w \in \mathscr{A}$, then multiplying the equation in (1.5) with $u - w$ and integrating by parts, we get

$$\int_\Omega \nabla u \cdot (\nabla u - \nabla w) = \int_\Omega f(u - w) dx$$

from this we obtain

$$\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx = \int_\Omega \nabla u \cdot \nabla w - \int_\Omega f w dx \leq \int_\Omega \frac{1}{2}(|\nabla u|^2 + |\nabla w|^2) dx - \int_\Omega f w dx$$

Therere $I(u) \leq I(w)$ for all $w \in \mathscr{A}$.
To prove the converse, if u solves the minimization problem (1.6). Then $t = 0$ is a critical point of the real valued function $i : \mathbb{R} \to \mathbb{R}$ given as

$$i(t) = I(u + tw)$$

$$i'(t) = \int_\Omega (\nabla u + t\nabla w)\nabla w dx - \int_\Omega f w dx$$

taking $t = 0$ we get

$$0 = \int_\Omega \nabla u \cdot \nabla w - \int_\Omega f w dx, \ \ \forall w \in \mathscr{A}.$$

Therefore u satisfies (1.5). □

Now consider the Neumann problem

$$-\Delta u = 0 \text{ in } \Omega \quad \frac{\partial u}{\partial n} = h(x) \ \partial\Omega. \tag{1.7}$$

Then it is necessary that $\int_{\partial\Omega} h(x) dS = 0$ (see exercise 16). Consider the functional I over $C^2(\overline{\Omega})$ as

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} h u dS.$$

and the minimization problem

$$E(u) = \min_{v \in C^2(\overline{\Omega})} E(v). \tag{1.8}$$

Theorem 3.1.3 *The problem* (1.7) *has a solution in* $C^2(\overline{\Omega})$ *if and only if* u *solves the problem in* (1.8).

Proof. Let u be a solution and let w be any other function. By taking $v = u - w$, we can show that

$$\begin{aligned} E(w) &= E(u) + \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} v\frac{\partial u}{\partial n} dS \\ &= E(u) + \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega v\Delta u dx \\ &\geq E(u). \end{aligned}$$

The conclusion follows from this. □

If we equip the space $\mathscr{A}$ with norm $(\int_\Omega |\nabla u|^2 dx)^{1/2}$, then the functional $I(u) = \|u\|^2 - \langle f, u\rangle$. In case of $\mathbb{R}^n$, if we take a functional

$$I(x) = \|x\|^2 - a \cdot x,\ a \in \mathbb{R}^n$$

Then it is easy to see that any minimizing sequence (x_n) that converges to $\min I(x)$ is bounded. By compactness, such there exists a convergent subsequence $x_{n_k} \to x_0$. Now it is easy to see that x_0 is the required minimizer. But this is not possible on $\mathscr{A}$ as it is infinite dimensional space. However taking a clue from above discussion we can try to find solution u_n in any finite dimensional subspace and then pass through the limit $n \to \infty$. Such methods leads to the so called Galerkin methods.
However there are some special domains for which we can solve the Elliptic equations by the separating the variables. For example the rectangle $[0,a] \times [0,b]$:

Example 3.1. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0,\ u(0,y) = u(a,y) = u(x,0) = 0,\ u(x,b) = f(x).$$

By assuming that the solution is in the form of $X(x)Y(y)$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = k$$

where is k is a constant. This leads to solving two ODEs

$$X'' = kX,\ Y'' = -kY$$

Also the boundary conditions

$$u(0,y) = u(a,y) = 0 \implies X(0) = 0, X(a) = 0.$$

Therefore solving the SLP:

$$X'' = kX,\ X(0) = X(a) = 0, \tag{1.9}$$

we get the solution

$$X(x) = \sin\frac{n\pi}{a}x,\ k = -\frac{n^2\pi^2}{a^2}, n = 1,2,3...$$

The second equation now becomes

$$Y'' = \frac{n^2\pi^2}{a^2}Y,\ Y(0) = 0$$

The general solution of this is

$$Y(y) = A\sinh\frac{n\pi}{a}y$$

where A is an arbitrary constant. Since these are solutions for each $n \in N$, we take

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$

Now the unknown constants A_n can be found by substituting the final boundary condition $u(x,b) = f(x)$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} b$$

Now using the orthogonality of the functions $\sin nx, n = 1,2,3....$ So A_n can be obtained from the relation:

$$A_n \sinh \frac{n\pi b}{a} \int_0^a \sin^2 \frac{n\pi x}{a} dx = \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

□

In case of Neumann problem, the boundary condition in SLP in (1.9), becomes

$$X'(0) = 0,\ X'(a) = 0.$$

Then the corresponding eigenvalues and eigen functions be taken in the infinite sum.
In the domains like disc $B_r(0)$, again one can write $u(x,y) = R(r)\Theta(\theta)$ where r, θ are poloar coordinates to solve the problems. These methods have limitation that the domain has to be a rectanlge or disc. In general it is not always possible to find the solutions exactly. To understand this we need to study qualitative properties of solutions with out knowing the solutions.

3.2 Maximum Principles

If we want to solve the Dirichlet problem in a general domain, we need first understant some qualitative properties of harmonic functions. In this direction we have the following maximum principle:

Theorem 3.2.1 Weak form: *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\Delta u \geq 0$ in Ω. Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$$

Proof. **Case 1:** $\Delta u > 0$ in Ω.
If $x_0 \in \Omega$ is a point of interior maximum of u. Then by second derivative test $\frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0$. Therefore, $\Delta u(x_0) \leq 0$. Contradiction to $\Delta u > 0$ in Ω.
Case 2: $\Delta u \geq 0$ in Ω.
Now consider the function $v(x) = u(x) + \varepsilon |x|^2$. Then

$$\Delta v(x) = \Delta u(x) + 2n\varepsilon > 0.$$

Then by case 1,

$$\max_{\overline{\Omega}} v(x) = \max_{\partial\Omega} v(x)$$

$$\max_{\overline{\Omega}} u + \varepsilon \min_{\overline{\Omega}} |x|^2 \leq \max_{\overline{\Omega}} (u(x) + \varepsilon |x|^2) = \max_{\overline{\Omega}} v(x) = \max_{\partial\Omega} v(x) \leq \max_{\partial\Omega} u(x) + \varepsilon \max_{\partial\Omega} |x|^2$$

Taking $\varepsilon \to 0$, conclusion follows.

Corollary 3.2.1 *If $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies $-\Delta u \geq 0$ in Ω, $u \geq 0$ on $\partial\Omega$. Then $u \geq 0$ in Ω.*

Remark 3.2.1 1. *In case of Harmonic functions, i.e., $\Delta u = 0$, above theorem holds for $-u$ as well. Therefore using $minu(x) = -\max(-u(x))$, we obtain*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

2. *If $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies $\Delta u = 0$ in Ω, then*

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

(follows from $|a| = \max(a,-a)$)

3. *If $u \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$. Then $u \equiv 0$ in $\overline{\Omega}$.*

Theorem 3.2.2 Uniqueness: *The Dirichlet problem $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ has at most one solution.*

Proof. Let u_1 and u_2 be two solutions, then by considering $u = u_1 - u_2$ we see that $-\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$. Therefore by the the (3) of Remark 1.1, we get $u = u_1 - u_2 \equiv 0$ in Ω.

Mean values: For a continuous function $h(x)$ on $\mathbb{R}^n$, let us introduce its spherical mean or average on a sphere of radius r and center x:

$$M_h(x,r) = \frac{1}{w_n}\int_{|\xi|=1} h(x+r\xi)\, dS_\xi,$$

where w_n denotes the area of the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and dS_ξ denotes surface measure.

Definition 3.2.1 Mean Value property: *A function $u \in C(\overline{\Omega})$ is said to satisfy Mean Value property if $u(\xi) = M_u(\xi,r)$ for all $r > 0$ such that $B_r(\xi) \subset\subset \Omega$.*

Theorem 3.2.3 Mean Value theorem (Gauss): *If $u \in C^2(\Omega)$ satisfies $\Delta u = 0$ in Ω then u satisfies mean value property.*

Proof. From (1.3), we get for $B_r(\xi) \subset \Omega$,

$$0 = \int_{B_r(\xi)} \Delta u dx = \int_{\partial B_r(\xi)} \frac{\partial u}{\partial n} dS = \int_{|x-\xi|=r} \frac{\partial u}{\partial n}(x) dS_x$$

Taking $x = \xi + ry$ with $|y| = 1$, $dS_x = r^{n-1} dS_y$ we have from above equation

$$0 = r^{n-1}\int_{|y|=1} \frac{\partial u}{\partial r}(\xi+ry) dS_y = r^{n-1}\frac{\partial}{\partial r}\int_{|y|=1} u(\xi+ry) dS_y = r^{n-1} w_n \frac{\partial}{\partial r} M_u(\xi,r)$$

That is $M_u(\xi,r)$ is independent of r. Therefore,

$$M_u(\xi,r) = \lim_{r\to 0} M_u(\xi,r) = u(\xi),$$

thanks to Lebesgue's theorem.

Theorem 3.2.4 *If $u \in C^2(\Omega)$ satisfies $\Delta u = 0$ in Ω. Then*

$$u(\xi) = \frac{n}{w_n} \int_{|x| \leq 1} u(\xi + rx) dx.$$

Proof. From the above theorem, we have

$$\begin{aligned} u(\xi) &= \frac{1}{w_n} \int_{|x|=1} u(\xi + rx) dS_x \\ &= \frac{1}{w_n} \int_{|x|=1} u(\xi + r\rho x) dS_x \end{aligned}$$

since the formula is true for any r. Multiplying both sides by ρ^{n-1} and integrating from 0 to 1, we get

$$\begin{aligned} \frac{u(\xi)}{n} = \int_0^1 \rho^{n-1} u(\xi) d\rho &= \frac{1}{w_n} \int_0^1 \rho^{n-1} \int_{|x|=1} u(\xi + r\rho x) dS_x d\rho \\ &= \frac{1}{w_n} \int_{|x| \leq 1} u(\xi + rx) dx. \end{aligned}$$

Corollary 3.2.2 *If $u \in C^2(\Omega)$ satisfies $\Delta u \geq 0$ in Ω. Then $u(\xi) \leq M_u(\xi, r)$.*

Proof. This follows from the previous theorem by replacing = by $\leq$

$$0 \leq \int_{B_r} \Delta u dx = \int_{\partial B_r} \frac{\partial u}{\partial n} dS = \ldots = \frac{\partial}{\partial r} M_u$$

That is $M_u(x, r)$ is increasing in r. Therefore

$$M_u(x, r) \geq M_u(x, 0) = u(\xi).$$

Corollary 3.2.3 *If $u \in C^2(\Omega)$ and $\Delta u \geq 0$ in Ω. Then*

$$u(\xi) \leq \frac{n}{w_n} \int_{|x| \leq 1} u(\xi + rx) dx.$$

Next we have the converse of Theorem 3.2.3:

Theorem 3.2.5 *If $u \in C^2(\Omega)$ satisfies meanvalue property then u is harmonic in Ω.*

Proof. Assume by contradiction that $\Delta u(\xi) > 0$ for some point ξ. Then there exists a ball B_ε around ξ such that $\Delta u > 0$ in $B_\varepsilon(\xi)$ and hence $\int_{B_\varepsilon(\xi)} \Delta u dx > 0$. But on the other hand, From the Theorem 1.1, we have

$$\int_{B_\varepsilon(\xi)} \Delta u dx = \varepsilon^{n-1} w_n \frac{\partial}{\partial r} M_u(\xi, r)$$

Now since u satisfies mean value property, then $M_u(\xi, r) = u(\xi)$. Therefore from the above equation we get

$$\int_{B_\varepsilon(\xi)} \Delta u dx = 0.$$

This is a contraction to $\Delta u > 0$ in $B_\varepsilon(\xi)$.

Next we have the following

Theorem 3.2.6 Strong form of Maximum principle
Let Ω be a connected domain and let $u \in C^2(\Omega)$ satisfies $\Delta u \geq 0$ in Ω, then either u is constant or $u(\xi) < \sup_{x\in\Omega} u(x)$ for all $\xi \in \Omega$.

Proof. Let $A = \sup_{x\in\Omega} u(x) < \infty$. By continuity of u, we know that the set

$$M = \{x \in \Omega : u(x) = A\}$$

is relatively closed in Ω. Now we claim that this is also open in Ω.

If $u(\xi) = A$. Then by above corollary, taking $\xi + rx = y$

$$u(\xi) \leq \frac{1}{w_n}\int_{|x|=1} u(\xi + rx)dS_x = \frac{1}{r^{n-1}w_n}\int_{|\xi-y|=r} u(y)dS_y.$$

Therefore,

$$0 \leq \int_{|\xi-y|=r} u(y)dS_y - w_n r^{n-1}A = \int_{|\xi-y|=r}(u(y)-A)dS_y.$$

But $u(y) - A \leq 0$ and u is continuous. Therefore $u(y) = A$ for all small ρ and $y \in \partial B_\rho(\xi)$. That is $y \in M$ for all y in a neighbourhood of ξ. Hence M is relatively open in Ω. Since Ω is connected, we get a contradiction. Therefore $\xi \notin \Omega^o$.

Corollary 3.2.4 *If Ω is connected domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u \geq 0$ in Ω. Then either u is constant OR $u(\xi) < \max_{\partial\Omega} u$ for all $\xi \in \Omega$.*

Corollary 3.2.5 Stability: *If $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u = f$ in Ω and $|u(x) - v(x)| \leq \varepsilon$ for all $x \in \partial\Omega$. Then*

$$|u(x) - v(x)| \leq \varepsilon \text{ for all } x \in \overline{\Omega}.$$

Moreover, one can show that if $u \in C(\Omega)$ is enough. In fact we have the following

Theorem 3.2.7 *Suppose that $u \in C(\Omega)$ has the mean value property. Then $u \in C^\infty(\Omega)$ and is harmonic.*

Proof. Let $\eta \in C_c^\infty(\Omega)$ be radially symmetric function with support in $B_\varepsilon(x) \subset\subset \Omega$ and $\int \eta = 1$. Then we claim that $(\eta * u)(x) = u(x)$ and $D^\alpha(\eta * u) = D^\alpha \eta * u$.

$$\begin{aligned}
(\eta * u)(x) = \int_{|x-y|<\varepsilon} \eta(x-y)u(y)dy &= \int_{|z|<\varepsilon} \eta(z)u(x-z)dz \\
&= \int_0^\varepsilon \left(\int_{\partial B_1(0)} \eta(r|z|)u(x-rz)dS(z)\right) r^{n-1}dr \\
&= \int_0^\varepsilon \left(\int_{\partial B_1(0)} u(x-rz)dS(z)\right) r^{n-1}\eta(r)dr \\
&= w_n u(x)\int_0^\varepsilon \eta(r)r^{n-1}dr \\
&= u(x)\int_{B_\varepsilon(x)} \eta(y)dy = u(x).
\end{aligned}$$

Next we will show that $D^\alpha(\eta * u) = D^\alpha * u$:

$$\frac{\partial}{\partial x_i}(\eta * u)(x) = \lim_{h\to 0}\int_{\mathbb{R}^n} \frac{[\eta(x-y-he_i) - \eta(x-y)]\,u(y)}{h}dy$$

Now since η is C^∞ function with compact support, we have

$$\frac{\eta(x-y-he_i)-\eta(x-y)}{h} \to \frac{\partial \eta}{\partial x_i}|(x-y) \text{ uniformly in } y.$$

Hence,

$$\frac{\partial}{\partial x_i}(\eta * u)(x) = \int_{\mathbb{R}^n} \frac{\partial \eta}{\partial x_i}(x-y)u(y)dy = (\frac{\partial \eta}{\partial x_i} * u)(x).$$

This concludes the proof.

Remark 3.2.2 *The above theorems imply that if u satisfies mean value property then it is harmonic and $u \in C^\infty(\Omega)$. If u is harmonic then it satisfies mean value property and hence is $C^\infty(\Omega)$.*

More generally, the following theorem is due to **Weyl**.

Theorem 3.2.8 (Weyl)*:*
Let $u : \Omega \to \mathbb{R}$ be measurable and locally integrable in Ω. If u satisfies $\Delta u = 0$ in $\mathscr{D}'$, in the sense of distributions,. Then u is harmonic and $C^\infty(\Omega)$.

Next we have the following Hopf maximum principle.

Definition 3.2.2 Interior ball condition *The boudary $\partial\Omega$ satisfies interior ball condition at x_0 if there is a ball $B_\varepsilon(x_1) \subset \Omega$ such that $\partial\Omega \cap \overline{B_\varepsilon(x_1)} = \{x_0\}$*

Theorem 3.2.9 (Hopf, Oleinik): *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\Delta u \geq 0$ in Ω and let Ω be a domain with its boundary $\partial\Omega$ satisfying interior ball condition at x^0 where x^0 is the point of maximum of u such that $u(x^0) > u(x)$ for all $x \in \Omega$. Then u is either constant OR $\frac{\partial u}{\partial n}(x^0) > 0$.*

Proof. Let us assume that there is a ball $B_r \subset \Omega$ such that $B_r \cap \partial\Omega = \{x^0\}$. Without loss of generality, let $B_r = B_r(0)$. Consider the function $v = e^{-\lambda|x|^2} - e^{-\lambda r^2}$, $x \in B_r(0)$. Then we can check that

$$v_{x_i} = e^{-\lambda|x|^2}(-2\lambda x_i)$$

Let $R = B_r(0) \backslash B_{r/2}(0)$ and let λ be fixed as large so that

$$\begin{aligned}\Delta v &= e^{-\lambda|x|^2}(4\lambda^2 \sum x_i^2 - 2\lambda) \\ &= e^{-\lambda|x|^2}(4\lambda^2|x|^2 - 2n\lambda) > 0 \text{ in } R.\end{aligned}$$

Now $\Delta(u+\varepsilon v - u(x^0)) \geq 0$ in R and $u+\varepsilon v - u(x^0) \leq 0$ on ∂R for ε small. By weak maximum principle,

$$u+\varepsilon v - u(x^0) \leq 0 \text{ in } R$$

and since $v(x^0) = 0$

$$u(x^0)+\varepsilon v(x^0) - u(x^0) = 0.$$

Therefore $\frac{\partial}{\partial n}(u+\varepsilon v)\Big|_{x^0} \geq 0$. Hence

$$\frac{\partial u}{\partial n}(x^0) \geq -\varepsilon\frac{\partial v}{\partial n}(x^0) = -\frac{\partial v}{\partial x_i}\frac{x_i}{r} = 2\lambda|x^0|^2 e^{-\lambda|x^0|^2} > 0.$$

3.3 Distributions

In this chapter we study briefly the thoery of Distributions. Let Ω be a open subset of $\mathbb{R}^n$. Then we define the function space

$$\mathscr{D}(\Omega) = \{u \in C^\infty(\Omega) \ : \ \text{support of } u \text{ is a compact subset of } \ \Omega\}$$

Example 3.2. The function

$$\phi(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}} & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

is $C^\infty(\mathbb{R}^n)$ with support equal to $B_1(0)$.

We defin the topology of $\mathscr{D}$ by convergent sequences

Definition 3.3.1 *A sequence $\{u_n\} \subset \mathscr{D}(\Omega)$ converges to 0 in the topology of $\mathscr{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that support$(u_n) \subset K$ and $\{D^\alpha u_n\}$ converges uniformly to 0 for all multi index α.*

Example 3.3. The sequence $\{\phi_n\}$ defined as $\phi_n(x) = \frac{1}{n}\phi(x)$ converges to 0 in $\mathscr{D}(\mathbb{R}^n)$. Indeed the supoort of $\phi_n(x) = B_1(0)$ and for all x,

$$|\phi_n(x)| \leq \frac{1}{n} \longrightarrow 0 \text{ as } n \to \infty.$$

Example 3.4. The sequence $\{\phi_n\}$ defined as $\phi_n(x) = \frac{1}{n}\phi(nx)$ does NOT converge in $\mathscr{D}(\mathbb{R}^n)$. The support of $\phi_n = B_{\frac{1}{n}}(0) \subset \overline{B_1(0)} := K$ and $\{\phi_n(x)\}$ converges uniformly to 0. But $\{D^\alpha \phi_n\}$ for $|\alpha| \geq 2$ does not converge.

Example 3.5. The sequence $\{\phi_n(x) = \frac{1}{n}\phi(x+n\overline{1})\}$. $\overline{1} = (1,1,...,1)$ does NOT converge in $\mathscr{D}(\mathbb{R}^n)$. In this case $\{D^\alpha \phi_n\}$ converges to 0 uniformly for all α. But the support of $\phi_n = B_1(n\overline{1})$ which is NOT contained in one compact set.

Definition 3.3.2 *A function $T : \mathscr{D}(\Omega) \to \mathbb{R}$ is called distribution if it is linear and continuous. That is T is an element of the dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$.*

Example 3.6. **Regular distribution:** If $f \in L^1_{loc}(\Omega)$. Then the distribution generated by f is defined as

$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx, \text{ for all } \phi \in \mathscr{D}(\Omega).$$

In fact, if K is the support of ϕ, then

$$|T_f(\phi)| \leq \|\phi\|_\infty \int_K |f(x)|dx$$

If $\phi_n \to 0$ in $\mathscr{D}(\Omega)$. Then there exists a compact set $K \subset \Omega$ such that support$(\phi_n) \subset K$ and $D^\alpha \phi_n \to 0$ uniformly on K. Therefore,

$$|T_f(\phi_n)| \leq \|\phi_n\|_\infty \int_K |f(x)|dx \leq C\|\phi_n\|_\infty \to 0 \text{ as } n \to \infty.$$

Example 3.7. Let μ be a Borel measure such that $\mu(K) < \infty$ for any compact subset. Then μ defines a distribution

$$T_\mu(\phi) = \int_\Omega \phi d\mu, \text{ for all } \phi \in \mathscr{D}(\Omega).$$

If $\phi_n \to 0$ in $\mathscr{D}(\Omega)$. Then Then there exists a compact set $K \subset \Omega$ such that support$(\phi_n) \subset K$ and $D^\alpha \phi_n \to 0$ uniformly on K.

$$|T_\mu(\phi_n)| \leq \|\phi_n\|_\infty |\int_K d\mu| = |\mu(K)| \|\phi_n\|_\infty \to 0 \text{ as } n \to \infty.$$

Remark 3.3.1 *The distribution generated by measure need not be a regular distribution. For example,*

Example 3.8. For any $x_0 \in \Omega$, define

$$T(\phi) = \phi(x_0), \text{ for all } \phi \in \mathscr{D}(\Omega).$$

Then it is easy to check T defines a distribution. This is also denoted as δ_{x_0} and is called Dirac delta distribution.

Theorem 3.3.1 *Let $f, g \in L^1_{loc}(\Omega)$. Then*

$$f \equiv g \iff T_f = T_g$$

Proof. It is obvious that if $f = g$ then $T_f = T_g$. For the converse if $T_f = T_g$. Then we have

$$0 = T_f(\phi) - T_g(\phi) = \int_\Omega (f-g)\phi dx, \text{ for all } \phi \in \mathscr{D}(\Omega).$$

Now the result follows from the following lemma. □

Lemma 3.3.1 *For $f \in L^1_{loc}(\Omega)$, if $\int_\Omega f(x)\phi(x)dx = 0$ for all $\phi \in \mathscr{D}(\Omega)$. Then $f(x) = 0$ a.e. in Ω.*

Next we have the following important theorem

Theorem 3.3.2 *For $T \in \mathscr{D}(\mathbb{R})$, if $\frac{dT}{dx} = 0$ then $T = T_C$ distribution generated by constant C.*

Calculus of Distributions Suppose ϕ and f are smooth functions with compact support in Ω. Then by integration by parts formula, we have

$$\int_\Omega \frac{\partial f}{\partial x_i}(x)\phi(x) = -\int_\Omega f(x)\frac{\partial \phi}{\partial x_i} dx$$

Motivated from this, we define

Definition 3.3.3 *The derivative $\frac{\partial T}{\partial x_i}$ of a distribution T is also a distribution defined as*

$$\frac{\partial T}{\partial x_i}(\phi) = -T(\frac{\partial \phi}{\partial x_i}), \textit{ for all } \phi \in \mathscr{D}(\Omega).$$

For any multi index α,

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \textit{ for all } \phi \in \mathscr{D}(\Omega).$$

Definition 3.3.4 Multiplication by functions*: For $\psi \in C^\infty(\Omega)$, ψT is a distribution defined by*

$$(\psi T)(\phi) = T(\psi \phi), \textit{ for all } \phi \in \mathscr{D}(\Omega).$$

Lemma 3.3.2 *(Product rule): For $\psi \in C^\infty$ and $T \in \mathscr{D}(\Omega)$ we have the following*

$$\frac{\partial}{\partial x_i}(\psi T) = \frac{\partial \psi}{\partial x_i} T + \psi \frac{\partial T}{\partial x_i}$$

Proof. Let $\phi \in \mathscr{D}(\Omega)$. Then for all $\phi \in \mathscr{D}(\Omega)$

$$\begin{aligned}
\psi \frac{\partial T}{\partial x_i}(\phi) &= \frac{\partial T}{\partial x_i}(\psi\phi) \\
&= -T(\frac{\partial}{\partial x_i}(\phi\psi)) = -T(\psi \frac{\partial \phi}{\partial x_i} + \frac{\partial \psi}{\partial x_i}\phi) \\
&= -(\psi T)(\frac{\partial \phi}{\partial x_i}) - (\frac{\partial \psi}{\partial x_i} T)(\phi) \\
&= [\frac{\partial}{\partial x_i}(\psi T)](\phi) - (\frac{\partial \psi}{\partial x_i} T)(\phi).
\end{aligned}$$

Hence the conclusion follows □

3.4 Fundamental solution

A fundamental solution $K(x)$ for the Laplace operator is a "distribution" satisfying the relation

$$\Delta K(x) = \delta(x) \ \text{ in } \mathscr{D}'(\mathbb{R}^n),$$

where δ is the delta distribution supported at $x = 0$. That is, $K(x)$ is a "locally integrable" function that satisfies,

$$\int_{\mathbb{R}^n} K(x) \Delta \phi(x) dx = \phi(0) \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

To obtain such solutions, we first assume that $u(x) = v(r), r = |x|$. By chain rule,

$$u_{x_i} = v'(r)\frac{x_i}{r}, \ \ \Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Therefore solving for $\Delta u = 0$ we get

$$v(r) = \begin{cases} b \log r + c & n = 2, \\ \frac{b}{r^{n-2}} + c, & n \geq 3, \end{cases}$$

where b, c are constants of integration.

Definition 3.4.1 *For $x \not\equiv 0$, the function*

$$\Phi(x) = \begin{cases} \frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{(2-n)w_n} |x|^{2-n} & n \geq 3. \end{cases}$$

where w_n is the surface area of the unit ball in $\mathbb{R}^n$ is called the fundamental solution of Laplacian operator.

Poisson's equation: The equation

$$-\Delta u = f(x) \text{ in } \mathbb{R}^n.$$

is known as Poisson equation. Here the problem is to find u for a given f. From the theory of distributions, we know that $(\Phi * f)(x)$ is a distributional solution of Poisson equation if f is a regular distribution with compact support. Now we will show that this is actually classical solution if we assume more regularity on the function $f(x)$.

From the construction above we see that $x \mapsto \Phi(x)$ is harmonic for all $x \neq 0$. If we shift the orgin to another point y the PDE remains the same. Also the function $x \mapsto \Phi(x-y)$ is also harmonic as a function of x, $x \neq y$. Now for a function $f(x)$ the mapping $x \mapsto \Phi(x-y)f(y), (x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$. We will show that the convolution

$$(\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

solves the problem. The important observation here is "differentiation under integral sign" is not allowed. Otherwise $\Delta(\Phi * f) = \int \Delta_x \Phi(x-y) f(y) = 0$. The reason is the second derivative of Φ is not integrable near 0.

Theorem 3.4.1 *Assume that $f \in C_c^2(\mathbb{R}^n)$ and let $u(x) = (\Phi * f)(x)$. Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u(x) = f(x)$ in $\mathbb{R}^n$.*

Proof. First note that $(\Phi * f)(x) = (f * \Phi)(x) = \int_{y \in supp(f)} \Phi(x-y) f(y) dy$ and so

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left(\frac{f(x-y+he_i) - f(x-y)}{h} \right) dy$$

Also since $f \in C_c^2(\mathbb{R}^n)$, $\frac{f(x-y+he_i)-f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y)$ uniformly. Therefore,

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y) dy, \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) dy.$$

Therefore,

$$\begin{aligned} \Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y) dy = \int_{\mathbb{R}^n} \Phi(y) \Delta_y f(x-y) dy \\ &= \int_{B_\varepsilon(0)} + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\Phi(y) \Delta_y f(x-y) dy) \\ &= I_\varepsilon + J_\varepsilon \end{aligned}$$

We can estimate I_ε and J_ε as follows

$$\begin{aligned} |I_\varepsilon| &\leq \|\Delta_y f\|_{L^\infty} \int_{B_\varepsilon(0)} \Phi(y) dy \\ &\leq \begin{cases} C\|\Delta f\|_{L^\infty} \int_0^\varepsilon \int_{S^{n-1}} |y|^{2-n} r^{n-1} r dr d\theta & n \geq 3 \\ C\|\Delta f\|_{L^\infty} \int_0^\varepsilon \int_0^{2\pi} (\log r) r dr d\theta & n = 2 \end{cases} \\ &= \begin{cases} o(\varepsilon^2) & n \geq 3 \\ o(\varepsilon^2 \log \varepsilon) & n = 2 \end{cases} \\ &\to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Integration by parts on J_ε, we get

$$\begin{aligned}
J_\varepsilon &= \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \frac{\partial^2 f}{\partial y_i^2}(x-y)dy \\
&= \sum_{i=1}^{n} \int_{\partial B_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial y_i}(x-y) n_i dS_y - \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \frac{\partial \Phi}{\partial y_i} \frac{\partial f}{\partial y_i}(x-y)dy \\
&= L_\varepsilon + K_\varepsilon
\end{aligned}$$

As above, we can estimate L_ε as follows

$$\begin{aligned}
|L_\varepsilon| &\le \|\nabla f\|_{L^\infty} \int_{\partial B_\varepsilon(0)} |\Phi(y)| dS_y \\
&\le \begin{cases} C \int_{\partial B_\varepsilon(0)} \varepsilon^{2-n} d\theta = C\varepsilon & n \ge 3 \\ C \int_{\partial B_\varepsilon(0)} \log|\varepsilon| d\theta = C(\log|\varepsilon|)(2\pi\varepsilon) & n = 2 \end{cases} \\
&\to 0 \; as \; \varepsilon \to 0.
\end{aligned}$$

To estimate K_ε, we again use integration by parts and using the fact that $\Delta\Phi = 0$ away from 0, we get

$$\begin{aligned}
K_\varepsilon &= -\sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \frac{\partial \Phi}{\partial y_i} \frac{\partial f}{\partial y_i}(x-y)dy \\
&= -\sum_{i=1}^{n} \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial y_i} f(x-y) n_i dS_y - \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Phi(y) f(x-y)dy \\
&= -\sum_{i=1}^{n} \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial y_i} f(x-y) n_i dS_y
\end{aligned}$$

and using the definition of Φ and $n = -\frac{y}{|y|}$ on ∂B_ε,

$$\frac{\partial \Phi}{\partial y_i} n_i = \frac{\partial \Phi}{\partial y_i} \frac{(-y_i)}{r} = \frac{1}{(n-2)w_n}(2-n)|y|^{1-n} \frac{y_i}{|y|} \frac{y_i}{r} = \frac{-1}{w_n}|y|^{1-n} = -\frac{\varepsilon^{1-n}}{w_n}$$

Therefore,

$$K_\varepsilon = \frac{-1}{w_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)dy \to -f(x), \text{ as } \varepsilon \to 0.$$

Therefore $\Delta u = \lim_{\varepsilon \to 0} (o(\varepsilon \log \varepsilon) + K_\varepsilon) = -f(x)$.

From the above theorem we infer that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) \Delta u(y) dy \text{ whenever } u \in C_0^\infty(\mathbb{R}^n).$$

The following result generalizes this formula allowing boundary terms.

Theorem 3.4.2 *If Ω is a smooth bounded domain in $\mathbb{R}^n$, $u \in C^2(\overline{\Omega})$, and $x \in \Omega$, then*

$$u(x) = \int_\Omega \Phi(x-y)\Delta u(y)dy + \int_{\partial\Omega} \left(u(y) \frac{\partial \Phi(x-y)}{\partial n_y} - \Phi(x-y) \frac{\partial u(y)}{\partial n} \right) dS_y. \tag{4.10}$$

Moregenerally, (4.10) *holds if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and the integral over Ω converges.*

Proof. Recall the Green identity (1.2):

$$\int_{\partial\Omega}(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n})dS=\int_{\Omega}(v\Delta u-u\Delta v)dx.$$

For $u\in C^2(\overline{\Omega})$ and $x\in\Omega$, $0<\varepsilon<dist(x,\partial\Omega)$, let $\Omega_\varepsilon=\Omega\backslash\overline{B_\varepsilon(x)}$. Applying the above identity on Ω_ε we get

$$\int_{\Omega_\varepsilon}\Phi(x-y)\Delta u(y)dy=\int_{\partial\Omega_e}\left(\Phi(x-y)\frac{\partial u}{\partial n}-u(y)\frac{\partial\Phi}{\partial n}(x-y)\right)dS_y+\int_{\Omega_\varepsilon}u\Delta\Phi(x-y)dy \tag{4.11}$$

Now note that $\Delta\Phi=0$ in Ω_ε and using the fact that $\Phi(x-y)\in L^1(\Omega)$, we get

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}\Phi(x-y)\Delta u(y)dy=\int_{\Omega}\Phi(x-y)\Delta u(y)dy$$

The boundary of Ω_ε consists of $\partial\Omega$ and $\partial B_\varepsilon(0)$:

$$\int_{\partial\Omega_\varepsilon}=\int_{\partial\Omega}+\int_{\partial B_\varepsilon(x)}$$

$$\left|\int_{|x-y|=\varepsilon}\Phi(x-y)\frac{\partial u}{\partial n}dS\right|\le\begin{cases}\|u\|_{C_1}\int_{|x-y|=\varepsilon}\varepsilon^{2-n}dS & n\ge3\\ \|u\|_{C^1}2\pi\varepsilon(|\log\varepsilon|) & n=2\end{cases}\longrightarrow 0\ as\ \varepsilon\to0.$$

$$\begin{aligned}\int_{|x-y|=\varepsilon}u(y)\frac{\partial\Phi}{\partial\nu}(x-y)dS_y&=\begin{cases}\frac{1}{(n-2)w_n}\int_{|x-y|=\varepsilon}u(y)\sum_{i=1}^{n}\frac{\partial}{\partial y_i}(|x-y|^{2-n})\cdot-\frac{x_i-y_i}{\varepsilon} & n\ge3,\\ \frac{1}{2\pi}\int_{|x-y|=\varepsilon}u(y)\sum_{i=1}^{n}\frac{\partial}{\partial y_i}(\log|x-y|)\cdot-\frac{x_i-y_i}{\varepsilon}, & n=2,\end{cases}\\ &=\begin{cases}\frac{1}{w_n}\int_{|x-y|=\varepsilon}u(y)|x-y|^{1-n}\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{\varepsilon} & n\ge3\\ \frac{1}{2\pi}\int_{|x-y|=\varepsilon}u(y)\frac{1}{|x-y|}\sum\frac{(x_i-y_i)^2}{\varepsilon} & n=2.\end{cases}\\ &\longrightarrow u(x)\text{ as }\varepsilon\to0.\end{aligned}$$

3.4.1 Dirichlet Problem

From (4.10), if u is a harmonic function, we get

Theorem 3.4.3 *If Ω is a smooth bounded domain in $\mathbb{R}^n$, $u\in C^2(\overline{\Omega})$ is a harmonic function, and $x\in\Omega$, then*

$$u(x)=\int_{\partial\Omega}\left(u(y)\frac{\partial\Phi(x-y)}{\partial n_y}-\Phi(x-y)\frac{\partial u(y)}{\partial n}\right)dS_y. \tag{4.12}$$

Dirichlet Problem: Given f and g find u satisfying:

$$-\Delta u(x)=f(x)\text{ in }\Omega,\quad u(x)=g(x)\text{ on }\partial\Omega. \tag{4.13}$$

So we need to modify the fundamental solution Φ. We do this by adding "corrector" to Φ so that one of the term on the Right hand side of (4.12) is zero. To achieve this we take a harmonic function $w(x)$ and consider the function $G(x,y)=\Phi(x-y)+w_x(y)$. Then taking $v(y)=w_x(y)$ in the Green's identity (1.2), we get

$$\int_{\Omega} w\Delta u dy + \int_{\partial\Omega}\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) dS = 0.$$

Also from (4.10), we have

$$\int_{\Omega} \Phi(x-y)\Delta u(y)dy + \int_{\partial\Omega}\left(u(y)\frac{\partial \Phi}{\partial n_y}(x-y) - \Phi(x-y)\frac{\partial u}{\partial n_y}\right) dS_y = u(x)$$

Adding these two equations, we get

$$u(x) = \int_{\Omega} G(x,y)\Delta u(y) + \int_{\partial\Omega}\left(u(y)\frac{\partial G}{\partial n_y} - G(x,y)\frac{\partial u}{\partial n_y}\right) dS_y$$

where $G(x,y) = \Phi(x-y) + w_x(y)$ Now if we can choose $w_x(y)$ satisfying $G(x,y) = 0$ for all $y \in \partial\Omega$. That is,

$$\Delta w_x(y) = 0, \text{ in } \Omega, \;\; w_x(y) = -\Phi(x-y) \text{ for all } x \in \Omega,\; y \in \partial\Omega.$$

Then

$$u(x) = \int_{\Omega} G(x,y)\Delta u(y)dy + \int_{\partial\Omega} u(y)\frac{\partial G}{\partial n}(x,y)dS_y. \tag{4.14}$$

Therefore we have

Theorem 3.4.4 *If $u(x)$ solves the problem* (4.13). *Then*

$$u(x) = \int_{\Omega} G(x,y)f(y)dy + \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n}(x,y)dS_y. \tag{4.15}$$

Proof. follows from (4.14).

To prove the converse of the above theorem, we investigate the regularity of the integral in

$$u(x) = \int_{\Omega} \Phi(x-y)f(y)dy.$$

As mentioned in the beginning of the section, if f is locally integrable, by extending the function $f(x)$ to be zero outside Ω, we get a compactly supported function and is a distributional solution of the equation $-\Delta u = f(x)$ in $\mathscr{D}'(\Omega)$. The following theorem expresses ways in which additional regularity of f improves the regularity of u.

Theorem 3.4.5 *For a bounded domain Ω and $f \in L^1(\Omega)$ define $u(x)$ by*

$$u(x) = \int_{\Omega} \Phi(x-y)f(y)dy.$$

Then

1. *u is harmonic and C^∞ in $\mathbb{R}^n \backslash \overline{\Omega}$.*
2. *if f is bounded on Ω, then $u \in C^1(\mathbb{R}^n)$.*
3. *if $f \in C^1(\overline{\Omega})$, then $u \in C^2(\Omega)$.*

Proof. (1): For any $x \in \mathbb{R}^n \backslash \overline{\Omega}$, $\Phi(x-y)$ is a smooth function as $x - y \neq 0$ for any $y \in \Omega$. So we can differentite under integral sign and since Φ is harmonic we get $\Delta u(x) = 0$.

(2): for $j = 1, 2, ...n$ we define

$$u_j(x) = \int_\Omega \frac{\partial \Phi(x-y)}{\partial x_j} f(y)dy,$$

which is well defined because $\partial\Phi/\partial x_j$ is $O(|x-y|^{1-n})$ as $|x-y| \to 0$. We first approximate u by smooth functions as follows: Let $\eta \in C^\infty(\mathbb{R}^n)$ satisfy $\eta(t) = 0$ for $t < 1$, $\eta(t) = 1$ for $t > 2$, and $0 \le \eta(t) \le 1$ and $0 \le \eta'(t) \le 2$ for all t, we define

$$u_\varepsilon(x) = \int_\Omega \Phi_\varepsilon(x-y)f(y)dy, \;\; \Phi_\varepsilon(x) = \Phi(z)\eta\left(\frac{|z|}{\varepsilon}\right).$$

Then u_ε is smooth function as $\Phi_\varepsilon = 0$ for $|x-y| < \varepsilon$. It is not difficult to check that $u_\varepsilon \to u$ uniformly on compact subsets of $\mathbb{R}^n$. Indeed, if $x \in K$,

$$\begin{aligned}
|u_\varepsilon(x) - u(x)| &= \left|\int_\Omega \left(\Phi(x-y)\eta\left(\frac{|x-y|}{\varepsilon}\right) - \Phi(x-y)\right) f(y)dy\right| \\
&\le \|f\|_\infty \int_{|x-y|<2\varepsilon} \Phi(x-y)\left(1-\eta\left(\frac{|x-y|}{\varepsilon}\right)\right)dy \\
&\to 0 \;\text{ as } \varepsilon \to 0.
\end{aligned}$$

Here we used the fact that $\eta(\frac{|x-y|}{\varepsilon}) = 1$ for $|x-y| > 2\varepsilon$. Next we will show that $\frac{\partial u_\varepsilon}{\partial x_j} \to u_j$. For this,

$$u_j(x) - \frac{\partial u_\varepsilon}{\partial x_j}(x) = \int_{|x-y|<2\varepsilon} \frac{\partial}{\partial x_j}\left[\left(1-\eta(\frac{|x-y|}{\varepsilon})\right)\Phi(x-y)\right] f(y)dy \tag{4.16}$$

$$= \int_{|z|<2\varepsilon} \left[\frac{\eta'(|z|/\varepsilon)}{\varepsilon}\frac{z_j}{|z|}\Phi(z) - \left(1-\eta(\frac{|z|}{\varepsilon})\right)\frac{\partial\Phi(z)}{\partial z_j}\right] f(x-z)dz \tag{4.17}$$

Now using the facts:

1. $|\eta'(t)| \le 1$
2. $\displaystyle\frac{1}{\varepsilon}\int_{|z|<2\varepsilon} \Phi(z)dz = o(\varepsilon)$
3. $\displaystyle\int_{|z|<2\varepsilon} |\frac{\partial\Phi(z)}{\partial z_j}| \le 2\varepsilon,$

we see that $\frac{\partial u_\varepsilon}{\partial x_j}$ converges uniformly to $\frac{\partial u}{\partial x_j}$. Therefore $u \in C^1(\mathbb{R}^n)$.
(3): We let again $v = \frac{\partial u}{\partial x_j}$, then introduce

$$\begin{aligned}
v_k(x) &= \int_\Omega \frac{\partial^2}{\partial x_k \partial x_j}\Phi(x-y)f(y)dy \\
&= \int_\Omega \frac{\partial^2}{\partial x_k \partial x_j}\Phi(x-y)(f(y)-f(x))dy + f(x)\int_\Omega \frac{\partial^2}{\partial x_k x_j}\Phi(x-y)dy
\end{aligned} \tag{4.18}$$

By divergence theorem

$$\int_\Omega \frac{\partial^2}{\partial x_k x_j}\Phi(x-y)dy = -\int_{\partial\Omega} \frac{\partial}{\partial x_j}\Phi(x-y)n_k dS_y.$$

Again we use the smoothing trick

$$v_\varepsilon(x) = \int_\Omega \frac{\partial}{\partial x_j}\Phi(x-y)\eta(\frac{|x-y|}{\varepsilon})f(y)dy$$

and calculate,

$$\frac{\partial v_\varepsilon}{\partial x_k} = \int_\Omega \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\Phi(x-y)\eta(\frac{|x-y|}{\varepsilon})\right](f(y)-f(x))dy + f(x)\int_\Omega \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\Phi(x-y)\eta(\frac{|x-y|}{\varepsilon})\right]dy$$

Again using divergence theorem on the second term, we obtain

$$\begin{aligned}\int_\Omega \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\Phi(x-y)\eta(\frac{|x-y|}{\varepsilon})\right]dy &= -\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Phi(x-y)\eta\left(\frac{|x-y|}{\varepsilon}\right)n_k dS_y \\ &= -\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Phi(x-y)n_k dS_y. \qquad (4.19)\end{aligned}$$

provided $\varepsilon > 0$ small enough such that $2\varepsilon < dist(x,\partial\Omega)$, and hence $\eta(|x-y|/\varepsilon) = 1$ for all $y \in \partial\Omega$. Thus we have from (4.18) and (4.19)

$$v_k(x) - \frac{\partial}{\partial x_k}v_\varepsilon(x) = \int_{|x-y|<2\varepsilon}\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}\Phi(x-y)\left(1-\eta\left(\frac{|x-y|}{\varepsilon}\right)\right)\right](f(y)-f(x))dy,$$

since the boundary integrals cancel each other. Estimating as earlier and noting that $\frac{\partial}{\partial z_i}\Phi(z) = o(|z|^{1-n})$, we get

$$\begin{aligned}|v_k(x) - \frac{\partial}{\partial x_k}v_\varepsilon(x)| &\le o(\varepsilon) + \int_{|x-y|\le 2\varepsilon}\left|\frac{\partial^2}{\partial x_k\partial x_j}\Phi(x-y)\right||f(x)-f(y)|dy \\ &\le C\sup_{y\in\Omega}\frac{|f(x)-f(y)|}{|x-y|}\int_{|x-y|<2\varepsilon}|x-y|^{1-n}dy = o(\varepsilon)\end{aligned}$$

where $C(\varepsilon)\to 0$ as $\varepsilon \to 0$. But $f \in C^1(\overline{\Omega})$ ensures that the supremum is finite, so we conclude that $v \in C^1(\Omega)$ and hence $u \in C^2(\Omega)$.

Remark 3.4.1 *A close observation at the proof suggests that $f(x) \in C^{0,\alpha}(\overline{\Omega})$ is enough to get the C^2 regularity.*

The domain Green's function is of the form $\Phi(x-y) + w_x(y)$ where w is a harmonic function. So now we need to understand the regularity boundary intergral $\int_{\partial\Omega}\frac{\partial G}{\partial n}g(y)dS_y$.

Poisson Integral formula: We want to solve the problem of finding harmonic functions with prescribed boundary values. Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$: Given g, find u satisfyng

$$(P): \Delta u = 0 \text{ in } \Omega, \quad u(y) = g(y) \text{ on } \partial\Omega.$$

Theorem 3.4.6 *Consider the domain $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n, x_n > 0\}$ and let g is continuous and bounded. Then the solution of problem*

$$(P): \qquad \Delta u = 0, \text{ in } \mathbb{R}^n_+, \quad u(y) = g(y) \text{ on } \partial\mathbb{R}^n_+$$

is given by the Poission Integral formula:

$$u(x) = \int_{\partial\mathbb{R}^n_+} H(x,y)g(y)dy, \qquad (4.20)$$

where $H(x,y) = \dfrac{2x_n}{w_n|y-x|^n}$, $x \in \mathbb{R}^n, y \in \partial\mathbb{R}^n_+$ *is called Poisson Kernel.*

Proof. The proof is by constructing Green's function for the domain $\mathbb{R}^n_+$. We need to define the harmonic function $w_x(y)$ satisfying

$$\Delta w_x(y) = 0 \text{ in } \Omega,\ \ w_x(y) = -\Phi(x-y),\ y \in \partial\mathbb{R}^n.$$

We use the notation

$$\mathbb{R}^n_+ = \{(x', x_n) = (x_1, x_2, ..., x_n),\ x_n > 0\}.$$

For $x = (x', x_n) \in \mathbb{R}^n_+$, Define its reflextion $x^* = (x_1, x_2, ..., -x_n) \notin \mathbb{R}^n_+$. Then the function $\Phi(y-x^*)$ is harmonic for all $y \in \mathbb{R}^n_+$. Moreover, for $y = (y_1, ..., y_{n-1}, 0) \in \partial\mathbb{R}^n_+$, we have

$$|y-x| = |y-x^*|$$

Therefore, $\Phi(y-x) = \Phi(y-x^*)$. Hence taking $w_x(y) = -\Phi(y-x)$

$$G(x,y) = \Phi(y-x) - \Phi(y-x^*)$$

is the Green's function for $\mathbb{R}^n_+$. Now to solve the problem (P) we use the formula (4.15), we notice that for $y \in \partial\mathbb{R}^n_+,\ x \in \mathbb{R}^n_+$

$$\frac{\partial \Phi}{\partial y_n}(y-x) = \frac{y_n - x_n}{w_n}|y-x|^{-n},$$

$$\frac{\partial G(y,x)}{\partial n} = -\left(\frac{\partial \Phi}{\partial y_n}(y-x) - \frac{\partial \Phi}{\partial y_n}(y-x^*)\right) = -\left(\frac{y_n - x_n}{w_n}|y-x|^{-n} - \frac{y_n - x_n^*}{w_n}|y-x^*|^{-n}\right)$$

For $y \in \partial\mathbb{R}^n_+$, we have $|y-x| = |y-x^*|$ and $x_n^* = -x_n$. Therefore,

$$\begin{aligned}\frac{\partial G(y,x)}{\partial n} &= -\frac{|y-x|^{-n}}{w_n}(y_n - x_n - y_n - x_n)\\ &= \frac{2x_n}{w_n}|y-x|^{-n} := H(x,y)\end{aligned}$$

Hence by (4.20) we get

$$u(x) = \frac{2x_n}{w_n}\int_{\partial\mathbb{R}^n}\frac{g(y)}{|x-y|^n}dy$$

for $y \in \partial\mathbb{R}^n_+$. Now since $x \mapsto G(y,x)$ is harmonic for $x \neq y$, we have $x \mapsto H(y,x) = -\frac{\partial G}{\partial y_n}(y,x)$ is harmonic because $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$. Hence defining $u(x)$ as (4.20) and applying Laplacian on $u(x)$, for $x \in \mathbb{R}^n_+$,

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+}\Delta_x H(y,x)g(y)dS_y = 0.$$

Claim: $\lim_{x \to x^0} u(x) = g(x^0)$ for $x^0 \in \partial\mathbb{R}^n_+$.
to prove the claim, choose $x^0 \in \partial\mathbb{R}^n_+$, and let $\varepsilon > 0$. Choose $\delta > 0$ such that $|g(y) - g(x^0)| < \varepsilon$ whenever $|y-x^0| < \delta, y \in \partial\mathbb{R}^n_+$. Then if $x \in \mathbb{R}^n_+$ with $|x-x^0| < \frac{\delta}{2}$, we have

$$\begin{aligned}|u(x)-g(x^0)| &= \int_{\partial\mathbb{R}^n_+} H(y,x)(g(y)-g(x^0))dS_y \\ &\leq \int_{\partial\mathbb{R}^n_+\setminus B_\delta(x^0)} + \int_{\partial\mathbb{R}^n_+\cap B_\delta(x^0)} := J+I\end{aligned}$$

Now since H is integrable, we estimate I as

$$|I| \leq \varepsilon \int_{\partial\mathbb{R}^n_+} H(y,x)dy \to 0 \text{ as } \varepsilon \to 0.$$

For J, as $x \to x^0$, we have $|x-x^0| < \frac{\delta}{2}$ and $|y-x^0| \geq \delta$. Therefore,

$$|y-x^0| \leq |y-x| + |x-x^0| \leq |y-x| + \frac{\delta}{2} \leq |y-x| + \frac{1}{2}|y-x^0|.$$

from this we see that $\frac{\delta}{2} \leq \frac{1}{2}|y-x^0| \leq |y-x|$. Therefore, we can estimate J as

$$\begin{aligned}|J| &\leq 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+} H(y,x)dy \\ &\leq 2x_n\|g\|_{L^\infty}2^n \int_{\partial\mathbb{R}_n^+} |y-x^0|^{-n}dy = 2x_n\|g\|_{L^\infty}2^n \int_\delta^\infty r^{-n}r^{n-2}dr \to 0 \text{ as } x_n \to 0.\end{aligned}$$

Note that in the above proof we have taken $y \in \partial\mathbb{R}^n_+$.

Green's function for Disc $B_a(0)$:

Here again we use the reflection/inverse point to define the correction function $w(x)$. For $x \in B_a(0)$, define $x^* = \frac{a^2 x}{|x|^2}$. This point is called inverse point with respect to the boundary $|y| = a$. Note that

$$\begin{aligned}|y-x^*|^2 = (y-x^*)\cdot(y-x^*) &= |y|^2 - 2y\cdot x^* + |x^*|^2 \\ &= a^2 - 2y\cdot x^* + |x^*|^2 = a^2 - 2y\cdot\frac{a^2x}{|x|^2} + a^4\frac{|x|^2}{|x|^4} \\ &= \frac{a^2}{|x|^2}\left(|x|^2 - 2x\cdot y + |y|^2\right) = \frac{a^2}{|x|^2}|x-y|^2\end{aligned}$$

Therefore, $\frac{|y-x^*|}{|y-x|} = \frac{a^2}{|x|^2}$ for all y such that $|y| = a$. Now define

$$w_x(y) = -\Phi\left(\frac{|x|}{a}|y-x^*|\right)$$

This is an anlytic function for $y \in B_a(0)$ and moreover $w_x(y) = -\Phi(y-x)$ for $y \in \partial B_a(0)$. The unit normal in this case is $\hat{n} = \frac{y}{a}$. So $\frac{\partial G}{\partial n} = \hat{n}\cdot\nabla G$.

$$\frac{\partial G}{\partial y_i}(x,y) = \frac{\partial \Phi}{\partial y_i}(y-x) - \frac{\partial}{\partial y_i}\Phi\left(\frac{|x|}{a}(y-x^*)\right)$$

$$\frac{\partial \Phi}{\partial y_i}(y-x) = \frac{1}{w_n}\frac{y_i-x_i}{|y-x|^n}$$

$$\begin{aligned}
\frac{\partial \Phi}{\partial y_i}\left(\frac{|x|}{a}(y-x^*)\right) &= \frac{1}{w_n}\left(\frac{|x|}{a}\right)^{2-n}(y_i-x_i^*)\,|y-x^*|^{-n} \\
&= \frac{1}{w_n}\left(\frac{|x|}{a}\right)^{2}\left(\frac{|x|}{a}|y-x^*|\right)^{-n}(y_i-x_i^*) \\
&= \left(\frac{|x|}{a}\right)^{2}\frac{1}{w_n}|y-x|^{-n}\left(\frac{y_i|x^2-a^2x_i}{|x|^2}\right) \\
&= \frac{1}{a^2}\frac{1}{w_n}|y-x|^{-n}\left(y_i|x^2-a^2x_i\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial G}{\partial y_i} =& \frac{1}{w_n|y-x|^n}\left(y_i-x_i-\frac{1}{a^2}(y_i|x|^2-a^2x_i)\right) \\
=& \frac{y_i}{a^2w_n|y-x|^n}(a^2-|x|^2)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial G}{\partial n} =& \sum_{i=1}^{n}\frac{\partial G}{\partial y_i}\cdot\frac{y_i}{a} \\
=& \frac{1}{a^2w_n|y-x|^n}(a^2-|x|^2)\sum_{i=1}^{n}\frac{y_i^2}{a} \\
=& \frac{1}{aw_n|y-x|^n}(a^2-|x|^2) \quad \text{since} \quad |y|=a.
\end{aligned}$$

So from the formula (4.20) we obtain

$$u(x)=\frac{a^2-|x|^2}{w_n}\int_{\partial B_a(0)}\frac{g(y)}{|x-y|^n}dS_y$$

Now following the similar steps as in the previous case one can show

$$\lim_{x\to x_0, x\in B_a(0)} u(x)=g(x^0), \text{ for any point } x^0\in\partial B_a(0).$$

□

In case of general domains the following is useful.

Theorem 3.4.7 *If u is harmonic function and O is an orthogonal $n\times n$ matrix. Then $v(x)=u(Ox)$ is also harmonic function.*

Proof. Let $O=[o_{ij}]$. We can compute

$$D_iv(x)=\sum_{k=1}^{n}D_ku(Ox)o_{ki},$$

$$D_{ij}v(x)=\sum_{l=1}^{n}\sum_{k=1}^{n}D_{kl}u(Ox)o_{ki}o_{lj}.$$

Since $OO^T=I$. We have for all $k,l=1,...n$

$$\sum_{i=1}^{n} o_{ki} o_{li} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}.$$

Thus

$$\begin{aligned}
\Delta v(x) &= \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) o_{ki} o_{li} \\
&= \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) \left(\sum_{i=1}^{n} o_{ki} o_{li} \right) \\
&= \sum_{l=1}^{n} \sum_{k=1}^{n} D_{kl} u(Ox) \delta_{kl} = \Delta u(Ox) = 0.
\end{aligned}$$

From the above discussions we established the following:

Theorem 3.4.8 *Let Ω be a domain in $\mathbb{R}^n$ such that the Green's function $G(x,y) = \Phi(x,y) + w_x(y)$ exists. Then $u(x)$ defined in* (4.15) *is the unique solution to the Dirichlet Problem* (4.14) *when $f \in C^1(\overline{\Omega})$ with all its first oder partial derivatives are in $L^\infty(\Omega)$ and $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$.*

Proof. Proof now follows from the Theorem 3.4.5 and the above Poisson integral formula. This as left as an exercise.

3.4.2 Properties of harmonic functions

Theorem 3.4.9 *(Harnack's inequality): Suppose $u \in C^2(\Omega)$ is a non-negative harmonic function and let $\Omega_1 \subset\subset \Omega$ be bouned sub domain. Then there exists C_1 depending only on Ω_1 such that*

$$\sup_{\Omega_1} u \leq C_1 \inf_{\Omega_1} u.$$

Proof. From Poison integral formula for ball $B_a(0)$, we have

$$u(x) = \frac{a^2 - |x|^2}{a w_n} \int_{\partial B_a(0)} \frac{u(y)}{|x-y|^n} dS_y \tag{4.21}$$

Now using the fact that

$$a - |x| \leq |x - y| \leq a + |x|, \text{ for } |y| = a,\ x \in B_a(0),$$

and

$$u(0) = \frac{1}{w_n a^{n-1}} \int_{\partial B_a(0)} u(y) dS_y$$

from (4.21), we get

$$\begin{aligned}
u(x) &\leq \frac{a + |x|}{a w_n (a - |x|)^{n-1}} \int_{\partial B_a(0)} u(y) dS_y \\
&= \frac{a^{n-2}(a + |x|)}{(a - |x|)^{n-1}} u(0)
\end{aligned}$$

Similarily, we get

$$u(x) \geq \frac{a^{n-2}(a-|x|)}{(a+|x|)} u(0)$$

From this we can show that by taking supremum and infimum over $B_{\frac{a}{2}}$ we get

$$\sup_{B_{\frac{a}{2}(0)}} u(x) \leq C \inf_{B_{\frac{a}{2}(0)}} u(x).$$

Now for any relatively compact set Ω_1 use a finite cover of $a/2$ balls.

Theorem 3.4.10 *Liouville's Theorem:*
If $u(x)$ is bounded and harmonic in $\mathbb{R}^n$. Then $u(x)$ is a constant.

Proof. From (4.21), differentiating $H(x,y)$ with respect to x_j, we get

$$H_{x_j}(x,y) = \frac{1}{aw_n}\left(\frac{-2x_j}{|x-y|^n} + n(a^2-|x|^2)\frac{y_j-x_j}{|x-y|^{n+2}}\right)$$

Therefore

$$\begin{aligned} H_{x_j}(0,y) &= \frac{na}{w_n}\frac{y_j}{a^{n+2}} \\ &= \frac{ny_j}{w_n a^{n+1}} \end{aligned}$$

Therefore from (4.21), we get

$$\begin{aligned} |\frac{\partial u}{\partial x_j}(0)| &= \int_{\partial B_a(0)} H_{x_j}(0,y)u(y)dS_y \\ &= \frac{n}{w_n a^{n+1}}\left|\int_{|y|=a} y_j u(y)dS_y\right| \\ &\leq \frac{n}{a^{n+1}}a^n \sup_{|y|=a}|u(y)| \leq \frac{n}{a}\sup_{B_a}|u(y)| \end{aligned}$$

This can be proved for any ball around a point x. That is, we get a_k

$$|u_{x_j}(x)| \leq \frac{n}{a_k}\|u\|_{L^\infty(\Omega)} \tag{4.22}$$

This shows that u is infinitely differentiable. Now since u is harmonic in the whole of $\mathbb{R}^n$, taking $a_k \to \infty$ we get u is constant.

We can extend (4.22) to compact subset $\Omega_1 \subset \Omega$ in which u is harmonic. Let $d > 0$ be the distance of Ω_1 to $\partial\Omega$, pick an increasing sequence $d_k \to d$. We can apply (4.22) to each of $B_{d_k}(\xi)$ for every $\xi\Omega_1$:

$$|\frac{\partial u}{\partial \xi_j}(\xi)| \leq \frac{n}{d_k}\max_{\partial B_{d_k}(\xi)}|u(x)| \leq \frac{n}{d_k}\|u\|_{L^\infty(\Omega)}$$

Now taking maximum over ξ and taking limit in d_k we get

$$\max_{\Omega_1}|\frac{\partial u}{\partial \xi_j}(\xi)| \leq \frac{n}{d}\|u\|_{L^\infty} \tag{4.23}$$

Theorem 3.4.11 *If $u_k \in C^2(\Omega)$ is a uniformly bounded sequence of harmonic function in Ω and let $\Omega_1 \subset\subset \Omega$, then u_k contains a subsequence that converges uniformly on Ω_1. Moreover the limit is also harmonic function.*

Proof. Using (4.22) on u_{x_i} we get replacing a by $a/2$

$$\left|u_{x_i x_j}(x)\right| \le \left(\frac{2n}{a}\right)^2 \sup_{B_{a/2}} |u(y)|.$$

Taking supremum over B_a we get

$$\sup_{B_a} |u_{x_i x_j}(x)| \le \left(\frac{2n}{a}\right)^2 \sup_{\Omega} |u(x)|$$

following the argument as in (4.23), we get for any Ω_1, a compact subset of Ω:

$$\sup_{\Omega_1} |u_{x_i x_j}(x)| \le \left(\frac{2n}{d}\right)^2 \sup_{\Omega} |u(x)|$$

where d is the distance from Ω_1 to $\partial\Omega$ Then by Ascoli-Arzela theorem there exists a subsequence that converges uniformly. Then applying the above inequality to $u_k - u_l$ shows that the second order derivatives of the u_k also converge uniformly on $B_a(x)$. Then applying the above inequality to $u_m - u_n$ we get that for this sub sequence we have $\Delta u_k \to \Delta u$. But $\Delta u_k = 0$ implies $\Delta u = 0$. Moreover, u is C^∞.

Theorem 3.4.12 *If $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$.*

Proof. For $\xi \in \Omega$, choose a ball $B_a(\xi) \subset\subset \Omega$ and $u|_{\partial B_a(\xi)} := g \in C(\partial B_a(\xi))$. Then by the existence of Green's function, we can show that there exists $u_1 \in C^\infty(\Omega)$ such that

$$\Delta u_1 = 0 \ \text{ in } B_a(\xi), \quad u_1 = g \text{ on } \partial B_a(\xi).$$

But by uniqueness we have $u = u_1$ in $B_a(\xi)$. Therefore $u \in C^\infty(\Omega)$.

Theorem 3.4.13 *If $u \in C^2(\Omega)$ is harmonic, then u is real analytic in Ω.*

Proof. Pick a point $x_0 \in \Omega$ and $a > 0$ so that $\overline{B_a}(x_0) \subset \Omega$. For simplicity $x_0 = 0$. We want to show that for a_1 small, the Taylor series for u converges to u on the ball $\overline{B_{a_1}(0)}$. for small a_1. Let $a_1 = \varepsilon a$. Then the Remainder term in the Taylor's theorem

$$R_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(tx)}{\alpha!} x^\alpha, \ \text{ for some } 0 \le t \le 1.$$

Let $M = \sup_{B_a} |u(x)|$. Then following (4.23),

$$|D^\alpha u(tx)| \le M \left(\frac{Nn}{(1-\varepsilon)a}\right)^N$$

Now using the estimate $|\alpha|^{|\alpha|} \le C\alpha!(en)^{|\alpha|}$ and $|x^\alpha| \le |x|^{|\alpha|}$, we get

$$\left|\frac{1}{\alpha!}|D^\alpha u(tx)|x^\alpha\right| \le CM\frac{e^N n^N}{n^N}\left[\frac{nN}{(1-\varepsilon)a}\right]^N (\varepsilon a)^N$$
$$= C\,M\frac{n^{2N}e^N a^N \varepsilon^N}{(1-\varepsilon)^N a^N}$$
$$= C\,M\left(\frac{n^2 e\varepsilon}{1-\varepsilon}\right)^N$$

Now we can choose ε small so that $\frac{\varepsilon e n^2}{1-\varepsilon} < \frac{1}{2}$. So

$$|R_N(x)| \le \sum_{|\alpha|=N} CM2^{-N} \le CM2^{-N}(N+1)^n \to 0 \; as\, N \to \infty.$$

That is, the Taylor's series converges. Hence u is real analytic.

3.5 Existence Theory: Perron's method

In this section we will prove the existence of harmonic function in general bounded domains (where Green's function is difficult to evaluate explicitely) with prescribed boundary values $u = g$ on the boundary. Motivated from the Corollary 3.2.2, we have the following

Definition 3.5.1 *A function $u \in C(\overline{\Omega})$ is called subharmonic if*

$$u(\xi) \le M_u(\xi, r)$$

for all $\xi \in \Omega$ and $B_r(\xi) \subset\subset \Omega$.

We note that the theorem 3.2.6, is indeed holds for any subharmonic functions:

Theorem 3.5.1 *Let Ω be a connected domain and let u be a subharmonic function. Then either u is constant or $u(\xi) < \sup_{x\in\Omega} u(x)$ for all $\xi \in \Omega$.*

Theorem 3.5.2 Comparison theorem: *Let u be a harmonic function with $u = g$ on $\partial\Omega$ and let v is subharmonic function in Ω with $v \le g$ on $\partial\Omega$. Then $v(x) \le u(x)$ in Ω.*

Proof. By Mean value theorem 3.2.3 we have

$$u(\xi) = M_u(\xi, r), \text{ and } v(\xi) \le M_v(\xi, r)$$

Therefore, $(v-u)(\xi) \le M_{v-u}(\xi, r)$. Hence $v-u$ is subharmonic function and $(v-u)(x) \le g(x) - g(x) = 0$ on the boundary. Therefore by the above theorem 3.5.1 we get $v \le u$ in Ω.

Therefore, we consider the class of functions

$$S_g = \left\{u \in C(\overline{\Omega}) : u \text{ is subharmonic in } \Omega \text{ and } u \le g \text{ on } \partial\Omega\right\}$$

Then S_g is nonempty for g bounded and continuous function in $\partial\Omega$. Indeed, let k be a constant such that $k \le \min_{\partial\Omega} g(x)$, then $k \in S_g$. Also, if $M = \max_{\partial\Omega}|g(y)|$, then for any $u \in S_g$, $u(x) \le M$. Moreover, $u - M \in S_0$. For $g \in C(\partial\Omega)$, define

$$w_g(x) = \sup\{u(x) \ : \ u \in S_g\}.$$

Then we have the following existence result

Theorem 3.5.3 *w_g is harmonic in Ω and $\lim_{x\to x_0} u(x) = g(x_0)$ for all $x_0 \in \partial\Omega$.*

Before we prove this, we need the following results:
For a subharmonic function u, define its harmonic representative as $\tilde{u}_{\xi,r}(x)$

$$\Delta \tilde{u}_{\xi,r} = 0 \text{ in } B_r(\xi), \ \ \tilde{u}_{\xi,r} = u \text{ on } \partial B_r(\xi), \quad B_r(\xi) \subset\subset \Omega$$

and its continuous extension $u_{\xi,r}(x)$

$$u_{\xi,r}(x) = \begin{cases} u(x) & x \in \Omega \backslash \overline{B_r(\xi)} \\ \tilde{u}_{\xi,r} & x \in B_r(\xi) \end{cases}$$

Lemma 3.5.1 *We have*

(a) $u(x) \leq u_{\xi,r}(x)$ *for all* $x \in \Omega$ *and*
(b) $u_{\xi,r}$ *is subharmonic in* Ω.

Proof. (a) $u - u_{\xi,r}$ is subharmonic in $B_r(\xi)$ and $u - u_{\xi,r} = 0$ on $\partial B_r(\xi)$. Then by theorem 3.5.2, $u(x) \leq u_{\xi,r}(x)$ in $B_r(\xi)$. Also $u(x) = u_{\xi,r}(x)$ for all $x \in \Omega \backslash \overline{B_r(\xi)}$.
(b) **Claim:** $u_{\xi,r}(x) \leq M_{u_{\xi,r}}(x,\rho)$ for all $x \in \Omega$ and $B_\rho(x) \subset\subset \Omega$.

(i) If $x \in B_r(\xi)$, then for small ρ such that, $B_\rho(x) \subset B_r(\xi)$, we have $u_{\xi,r}$ is harmonic in $B_r(\xi)$. Therefore claim holds.
(ii) If $x \in \Omega \backslash \overline{B_r(\xi)}$, then choose $B_\rho \subset \Omega \backslash \overline{B_r(\xi)}$ and u is subharmonic in Ω proves the claim.
(iii) If $x \in \partial B_r(\xi)$. Then for $x \in \Omega$ we have $u(x) \leq u_{\xi,r}(x)$. Therefore

$$M_u(x,\rho) \leq M_{u_{\xi,r}}(x,\rho) \tag{5.24}$$

Now $x \in \partial B_r(\xi)$ and u is subharmonic in Ω implies

$$u_{\xi,r}(x) = u(x) \leq M_u(x,\rho) \tag{5.25}$$

From (5.24) and (5.25), we get

$$u_{\xi,r}(x) = u(x) \leq M_u(x,\rho) \leq M_{u_{\xi,r}}(x,\rho)$$

Lemma 3.5.2 *If $u_1, u_2, ..., u_k$ are subharmonic in Ω, then $v = \max\{u_1, u_2, ...u_k\}$ is subharmonic in Ω.*

Proof.

$$u_j \text{ is subharmonic} \implies u_j(\xi) \leq M_{u_j}(\xi, r)$$

$$u_j \leq v \implies M_{u_j}(\xi,r) \leq M_v(\xi,r)$$

Therefore,

$$v(\xi) = \max\{u_1(xi), u_2(\xi), ...u_k(\xi)\} \leq \max\left\{M_{u_1}(\xi,r), M_{u_2}(\xi,r), ..., M_{u_k}(\xi,r)\right\} \leq M_v(\xi,r)\}$$

Hence v is subharmonic in Ω.

Lemma 3.5.3 *w_g is harmonic in Ω.*

Proof. It is enough to show that w_g is harmonic in $B_{\frac{r}{2}}(\xi) \subset \overline{B_r(\xi)} \subset \Omega$. The proof is divided into several steps:

1. By the definition of w_g, we can find functions $u_k^j \in S_g$ such that

$$w_g(\xi) = \lim_{j\to\infty} u^j(\xi)$$

2. Let $m = \min\{g(x) : x \in \partial\Omega\}$ and $M = \max\{g(x) : x \in \partial\Omega\}$. Then

$$m \le u^j(x) \le M, \text{ for all } x \in \Omega$$

3. We can replace u^j by $u^j_{\xi,r}$ in the limit in step 1 as $u^j \le u^j_{\xi,r}$. Still the limit will be same and so with out of loss of generality we may assume u^j in the limit in step 1 is harmonic in $B_r(\xi)$.
4. By step 2, $\{u^j\}$ is uniformly bounded and by step 3 $\{u^j\}$ is harmonic in B_r. Therefore u^j converges to a harmonic function $w(x)$ in $B_{r/2}(\xi)$.
5. $w(x) = w_g(x)$ for all x in $B_{r/2}(\xi)$. Since w_g is supremum, $w(x) \le w_g(x)$ for all $x \in B_{r/2}$. Now if there exists a point $x' \in B_{r/2}$ such that $w(x') < w_g(x')$. That means there exists $v \in S_g$ such that $w(x') < v(x') < w_g(x')$. Now replacing u^j by $\max\{u^j, v\}$ and its harmonic replacement, we obtain the limit η such that

$$w(x) \le \eta(x) \le w_g(x), \; for\, all\; x \in B_{r/2} \text{ and } w(\xi) = \eta(\xi)$$

On the other hand we also have $w(\xi) = w_g(\xi)$. Therefore

$$\frac{1}{|B_{r/2}|}\int_{B_{r/2}} \eta(y)dy \ge \frac{1}{|B_{r/2}|}\int_{B_{r/2}} w(y)dy = w(\xi) = \eta(\xi) = \frac{1}{|B_{r/2}|}\int_{B_{r/2}} \eta(y)dy$$

The only possibility is $w \equiv \eta$ in $B_{r/2}$. But this is a contradiction as

$$\eta(x') \ge v(x') > w(x').$$

Therefore $w \equiv w_g$ in $B_{r/2}$. □

To proceed further we need the following

Definition 3.5.2 Barrier function: *$Q_z \in C(\overline{\Omega})$ is called barrier function at $z \in \partial\Omega$ if*

1. *Q_z is subharmonic in Ω*
2. *$Q_z(z) = 0$*
3. *$Q_z(x) < 0$ for any $x \in \partial\Omega \setminus \{z\}$.*

Definition 3.5.3 *Regular boundary:*
A point $z \in \partial\Omega$ is regular if there exists a barrier function at z and $\partial\Omega$ is regular if all $z \in \partial\Omega$ are regular.

Example 3.9. If $\partial\Omega$ satisfies exterior sphere condition at $z \in \partial\Omega$, then z is a regular point. In this case if $B_\varepsilon(\xi)$ is exterior sphere at $z \in \partial\Omega$, then we can take

$$Q_z(x) = \begin{cases} -(\frac{|x-\xi|}{\varepsilon})^{2-n}, & n \ge 3 \\ -\log(\frac{|x-\xi|}{\varepsilon}) & n = 2 \end{cases}$$

Lemma 3.5.4 *If $g \in C(\partial\Omega)$ and $z \in \partial\Omega$ is regular then*

$$\lim_{x\to z} w_g(x) = g(z)$$

Proof. Consider the function $u_-(x) = g(z) - \varepsilon + kQ_z(x)$ (where ε, k will be chosen later). Then u_- is subharmonic in Ω.
Claim 1: $u_-(x) \leq g(x)$ for all $x \in \partial\Omega$.
By continuity of g, there exists $\delta > 0$ such that

$$|x-z| < \delta \implies |g(x) - g(z) < \varepsilon.$$

If $x \in \partial\Omega$ and $|x-z| < \delta$. Then

$$u_-(x) - g(x) = g(z) - g(x) - \varepsilon + kQ_z(x) < 0$$

If $x \in \partial\Omega,\ |x-z| \geq \delta$. Then choose k such that $kQ_z(x) \leq -2M$ where $M = \max_{\partial\Omega} |g(x)|$. Then again,

$$u_-(x) - g(x) \leq 2M - \varepsilon + kQ_z(x) < 0.$$

Therefore, $u_-(x) \leq g(x)$ on $\partial\Omega$ and u_- is subharmonic. Hence $u_- \leq g$ and

$$u_-(x) \leq w_g(x) \text{ for all} x \in \partial\Omega.$$

Similarly, defining

$$u_+(x) = g(z) + \varepsilon - kQ_z(x)$$

Then $u_+ \in C(\overline{\Omega})$ and $-u_+$ is subharmonic.
Claim 2: $-u_+ \leq -g$ on $\partial\Omega$.

$$|x-z| < \delta \implies -u_+(x) + g(x) = -g(z) + g(x) - \varepsilon + kQ_z(x) < 0$$

If $|x-z| \geq \delta$ as earlier choose k such that $kQ_z(x) \leq -2M$.

$$-u_+(x) + g(x) = -g(z) + g(x) - \varepsilon + kQ_z(x) \leq 2M - \varepsilon - 2M < 0.$$

Therefore, $-u_+(x) \leq -g(x)$ on $\partial\Omega$.
Therefore for any $u \in S_g$, $u - u_+$ is subharmonic and $u - u_+ \leq 0$ on $\partial\Omega$. Hence by Maximum principle, $u < u_+$ in Ω. Again by the definition of w_g, we get

$$w_g(x) \leq u_+(x) \text{ in } \Omega.$$

Hence from the cliams, we have

$$u_-(x) \leq w_g(x) \leq u_+(x) \text{ in } \Omega.$$

Now using the fact that $Q_z(x) \to 0$ as $x \to z$ we get $|w_g(x) - g(z)| \to 0$. □

3.6 Problems

1. Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies
$$\Delta u + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} - u \geq 0, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$
Then show that $u \leq 0$ in Ω.
2. Prove the weak form of maximum principle for general second order elliptic opeator
$$Lu = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}), \qquad \sum a_{ij}\xi_i\xi_j \geq C|\xi|^2, \xi \in \mathbb{R}^n$$
3. Suppose $q(x) \geq 0$ for $x \in \Omega$ and consider solutions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of $\Delta u - q(x)u = 0$ in Ω. Establish uniqueness theorem for Dirichlet problem.
4. If Ω is a bounded domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω, then show that
$$\max_{\overline{\Omega}} |u(x)| = \max_{\partial\Omega} |u(x)|.$$
5. Construct the Green's function for the domain
$$(a)\ \{(x,y) \in \mathbb{R}^2 :\ x > 0,\ y > 0\} \quad (b)\ \{(x,y) : x > 0,\ x^2 + y^2 < 1\}.$$
6. Show that $\Phi(x)$ is a distributional solutoin of $-\Delta u = \delta$ in $\mathbb{R}^n$.
7. Suppose $u \in C(\Omega)$ satisfies mean value property in Ω. Then show that boundary values on $\overline{B_r(\xi)} \subset \Omega$ uniquely determine u.
8. Show that the bounded solution of the Dirichlet problem in a half-space is unique. What can you say about unbounded solutions?
9. Let $\Omega = B_a(0)$, $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\Omega_0 = \{x \in \Omega, x_n = 0\}$. If $u \in C^2(\Omega) \cap C(\Omega_+ \cup \Omega_0)$ is harmonic in Ω_+, and $u = 0$ on Ω_0. Prove that u may be extended to a harmonic function on all of Ω (use reflection).
10. Show that $u \in C^2(\Omega)$ is subharmonic then $\Delta u \geq 0$ in Ω.
11. Let f, g be continuous and bounded functions and let u is a smooth solution of $-\Delta u = f$ in $B(0,1)$, $\quad u = g$ on $\partial B(0,1)$. Then Prove the following stability estimate: There exists a constant C, depending only on n, such that
$$\max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|)$$
12. Prove the stability estimate in problem 11 above for any bounded domain Ω instead of $B(0,1)$.
13. Let $\Omega = \mathbb{R}^n_+$ and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = 0$ on the boundary. If $u(x)$ is bounded then show that $u \equiv 0$. If u is not bounded, then show that there can be more than one such functions.
14. Let u be a solution of Neumann Problem
$$\Delta u = 0 \text{ in } \Omega \quad \frac{\partial u}{\partial n} = h(x) \text{ on } \partial\Omega.$$
Then show that $\int_{\partial\Omega} h(x)dS = 0$.