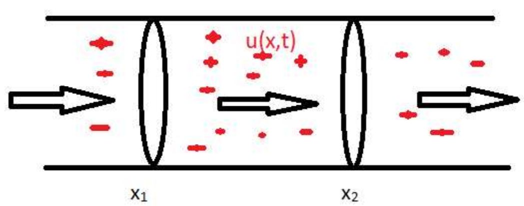
Chapter 1

First order Partial Differential Equations

1.1 Transport Equation

In this section we will study the transport equations and its solutions. Consider a fluid flowing through a thin straight tube whose cross section is A. Suppose fluid contain a contamination or some chemical or trafic of vehicles, whose concentration at position x at time t is $u(x,t)$. Then at time t, the amount of contaminant



Flow through a channel

in a section of tube between x_1 and x_2 is

$$\int_{x_1}^{x_2} u(x,t)A\,dx$$

Let $q(x,t)$ be the flux through the cross section x at time t. Then amount of contaminant that flows through a plane cross section at x during the time between t_1 and t_2 is

$$\int_{t_1}^{t_2} q(x,t)A\,dt$$

The equation of continuity:
The contaminant present at time t_2 = {The contaminant present at time t_1} + { the amount flows in from x_1 between time t_1 and t_2 } - { The contaminant that flows out at x_2 between times t_1 and t_2}. + { internally generated amount through internal sources.}
If we assume that there are no internal sources, this implies

$$\int_{x_1}^{x_2} u(x,t_2)A\,dx = \int_{x_1}^{x_2} u(x,t_1)A\,dx + \int_{t_1}^{t_2} q(x_1,t)A\,dt - \int_{t_1}^{t_2} q(x_2,t)A\,dt$$

That is,

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\partial_t u(x,t)A dt dx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\partial_x q(x,t)A dt$$

This is true for all intervals (x_1, x_2) and times t_1, t_2 if

$$\partial_t(uA) + \partial_x(qA) = 0.$$

The simplest case is when $q(x,t) = \alpha u(x,t)$ for some scalar $\alpha > 0$. The resultant equation known as transport equation:

$$u_t + \alpha u_x = 0.\ \alpha > 0,\ t \geq 0,\ x \in \mathbb{R}$$

.

Initial value problem: Consider the problem of finding $u(x,t)$ satisfying

$$u_t + a u_x = 0,\ t \geq 0,\ x \in \mathbb{R}$$
$$u(x,0) = h(x),\ x \in \mathbb{R}$$

where $a \in \mathbb{R}$ is a constant.

To understand this, let us first take $a = 0$. Then the equation is $u_t = 0$ by integration we find that the solution is

$$u(x,t) = c(x)$$

Now by imposing the initial condition we get $u(x,t) = h(x)$. That is to get a solution at the point (x,t) we project this point on to x axis and take the initial value at this point as the solution at (x,t). That is we are traveling back along the lines parallel to t-axis to the initial curve to identify the solution. Now we should think of what situation will lead to the lines/curves that are not parallel to t-axis?

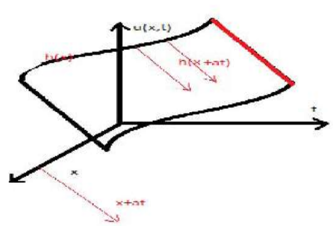
We see this equation as directional derivative of a function $u(x,t)$ in the direction $(1,a)$. So if we consider the function $g : \mathbb{R} \to \mathbb{R}^2$ and $F : \mathbb{R} \to \mathbb{R}$

$$g(t) = (x + at, t),\ F(t) = u(g(t)).$$

Then

$$F'(t) = u_x a + u_t = 0$$

Therefore $F(t) = F(0) = u(g(0)) = u(x,0)$. Therefore the solution $u(x,t)$ is $u(x,t) = h(x - at)$.



So we see that we can try to integrate along the directions along which the directional derivative is constant. Using this we can also solve non-homogeneous problem like

$$u_t + au_x = f(x,t),\ t \geq 0, x \in \mathbb{R}$$
$$u(x,0) = h(x)$$

In this case, we write the parametric form of lines with slope a as

$$x(s) = x + as,\ t(s) = t + s$$

represents a line in $\mathbb{R}^2 : s \to (x(s),t(s))$ passes through (x,t). Then define

$$F(s) = u(x(s),t(s))$$

We immediately see that

$$F'(s) = u_x\, x'(s) + u_t\, t'(s) = au_x + u_t = f(x(s),t(s)) = f(x+as,t+s)$$

This implies

$$F(0) - F(-t) = \int_{-t}^{0} f(x+as,t+s)ds = \int_{0}^{t} f(x+(s-t)a,s)ds$$

On the other hand

$$F(0) - F(-t) = u(x,t) - u(x(-t),t(-t)) = u(x,t) - u(x-at,0) = u(x,t) - h(x-at)$$

Therefore,

$$u(x,t) = h(x-at) + \int_{0}^{t} f(x+(s-t)a,s)ds,\ x \in \mathbb{R},\ t \geq 0.$$

1.2 Method of characteristics

Consider the partial differential equation in two variables x,y

$$au_x + bu_y = c \tag{2.1}$$

where a,b,c are continuous functions in x,y,u. Let $u(x,y)$ be the solution and let $z = u(x,y)$ be the graph of u. That is, the level surface $S : u(x,y) - z = 0$. Let $z_0 = u(x_0,y_0)$. The normal to this surface at a point (x_0,y_0,z_0) is $N = (u_x,u_y,-1)$ at (x_0,y_0).

The equation in (2.1) implies that the vector $V_0 = (a,b,c)$ is perpendicular to the normal N. Hence V_0 must lie on the tangent plane to the surface $z = u(x,y)$. So our aim is to find the surface $z = u(x,y)$ knowing that the vector field $\{a(x,y,z),b(x,y,z),c(x,y,z)\}$ lies on the tangent place. Such surface is called **Integral surface**.

Cauchy Problem: Given a curve Γ in $\mathbb{R}^3$ whose projection on the plane is γ, find a function $u(x,y)$ satisfying (2.1) such that the level surface $z = u(x,y)$ contains Γ. In other words, find $u(x,y)$ satisfying

$$\begin{aligned} au_x + bu_y &= c \ \text{ in } U \\ u(x,y) &= h(x,y) \ \text{ on } \gamma \end{aligned} \tag{2.2}$$

where U is open domain that contains the curve γ.

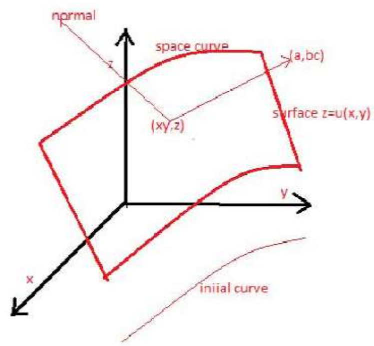


Fig. 1.1 Integral surface

Let us assume that the surface S is parametrizable and

$$S = \{(x(s,t), y(s,t), z(s,t))\}$$

with $t \geq 0$ and $s \in I \subset \mathbb{R}$. Also let the initial space curve is $\Gamma = \{(x(s,0), y(s,0), z(s,0))\}$. Therefore from the given initial condition $u(x,y) = h$ on γ implies $z(s,0) = h(x(s,0), y(s,0)) = h(s)$. We may assume that $x(s,0) = f(s)$ and $y(s,0) = g(s)$.
Since (a,b,c) is perpendicular to the normal, it is proportional to the tangent vector. So it is natural that it satisfies the system of equations with initial conditions

$$\begin{aligned} \frac{dx}{dt} &= a(x(s,t), y(s,t)z(s,t)), \ \ x(s,0) = f(s) \\ \frac{dy}{dt} &= b(x(s,t), y(s,t), z(s,t)), \ \ y(s,0) = g(s) \\ \frac{dz}{dt} &= c(x(s,t), y(s,t), z(s,t)), \ \ z(s,0) = h(s) \end{aligned}$$

We can solve this system of equations uniquely(Picard's theorem) for all small t under the assumption that (a,b,c) are C^1 functions. The curves so obtained are called characteristic curves. We will find the surface as union of these curves obtained from the above ODE system. Then at each point (x_0, y_0, z_0) the tangent plane contains the vector $V(x_0, y_0, z_0) = (a,b,c)$. In otherwords, smooth union of characteristic curves is the Integral surface.
Now to obtain the surface in the variables x, y we need to invert the map $(x,y) \to (s,t)$. For this we use **Inverse function theorem**. The above map is invertible near $t = 0$ if the Jacobian is not zero at $t = 0$. i.e.,

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} f'(s) & g'(s) \\ a & b \end{vmatrix} = b f'(s) - a g'(s) \neq 0 \text{ at } t = 0.$$

Definition 1.2.1 Non-characteristic curve: *A curve γ is called non-characteristic if $b\,f'(s) - a\,g'(s) \neq 0$.*

We proved the following

Theorem 1.2.1 *Let γ be a non-characteristic curve and let a, b, c are C^1 functions. Then the Cauchy problem in* (2.2) *has a solution in a neighbourhood of the initial curve γ.*

Example 1.1. Solve the Initial value problem:

$$2u_x + 3u_y = 0,\ (x,y) \in \mathbb{R}^2,\ \ u(x,0) = g(x),\ x \in \mathbb{R}.$$

The parametrization of initial curve $\gamma = \{(s, 0, g(s)),\ s \in \mathbb{R}\}$. The Jacobian is

$$J = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \neq 0.$$

So the solution exists near the initial curve γ. The characteristic equations are

$$\frac{dx}{dt} = 2,\ \ x(s,0) = s$$
$$\frac{dy}{dt} = 3,\ \ y(s,0) = 0$$
$$\frac{dz}{dt} = 0,\ \ z(s,0) = g(s)$$

The solution is

$$x(s,t) = 2t + s,\ y(s,t) = 3t,\ z(s,t) = g(s)$$

Inverting the variables we get

$$t = \frac{y}{3},\ \text{and}\ \ s = x - \frac{2}{3}y$$

Hence $u(x,y) = z = g(s) = g(x - \frac{2}{3}y)$.

Example 1.2. Characteristic problem:
Consider the problem $u_x + u_y = 0,\ \ u = g$ on γ. Then show that

(a) Solution exists uniquely if γ is not parallel to $y = x$.
(b) If γ is parallel to $y = x$, then solution exists if and only if g is constant.

Solution: (a): If γ is parallel to $y = x$, then $f'(s) = g'(s) = 1$ and the Jacobian

$$J = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

(b) : Let us solve the problem with $u(x,0) = g(x)$. Then

$$\frac{dx}{dt} = 1,\ \ x(s,0) = s$$
$$\frac{dy}{dt} = 1,\ \ y(s,0) = 0$$
$$\frac{dz}{dt} = 0,\ \ z(s,0) = g(s)$$

Then the solution is $x(s,t) = t+s,\ y(s,t) = t,\ z(s,t) = g(s)$. Hence

$$u(x,y) = z = g(x-y)$$

Therefore on $y = x$, $u(x,x) = g(0)$. That is, solution of the problem exists if and only $u(x,x) = g(x)$= constant. In this case note that $g(x-y)$ solves the equation $u_x + u_y = 0$ for any function g. If the initial condition is $u(x,x) = 5$ then $5+(x-y)^k, k \geq 0$ are all solutions. So the characteristic problem does not admit solution for all initial values. If solution exists, then there are infinitely many solutions.

1.2.1 Semi linear equations

The first order equation $au_x + bu_y = c$ is called semi linear equation if $a = a(x,y), b = b(x,y)$ and $c = c(x,y,u)$.

Example 1.3. Solve the Cauchy problem

$$u_x + 2u_y = u^2,\ x \in \mathbb{R},\ y > 0\ \ u(x,0) = h(x).$$

A parametrization of initial curve is $\{(s,0,h(s)),\ s \in \mathbb{R}\}$. The characteristic equations are

$$\begin{aligned}
\frac{dx}{dt} &= 1,\ \ x(s,0) = s \\
\frac{dy}{dt} &= 2,\ \ y(s,0) = 0 \\
\frac{dz}{dt} &= z^2,\ \ z(s,0) = g(s)
\end{aligned}$$

The Jacobian J is

$$J = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \neq 0.$$

Therefore

$$y = 2t,\ x = t+s,\ \text{and}\ \frac{-1}{z} = t - \frac{1}{h(s)}$$

Inverting the variables, we get

$$t = \frac{y}{2},\ s = x - \frac{y}{2},\ \text{and}\ z = \frac{h(s)}{1-th(s)}.$$

The characteristic lines are $y = 2x+2s$ and the solution is

$$u(x,y) = z = \frac{h(x-\frac{y}{2})}{1-\frac{y}{2}h(x-\frac{y}{2}})$$

which is defined upto $1 - \dfrac{y}{2}h(x-\dfrac{y}{2}) \neq 0.$ □

Example 1.4. **Characteristic problem:** Consider the initial problem

$$u_x + xu_y = u^2,\ x,y \in \mathbb{R},\ \ u\left(x, \frac{x^2}{2}\right) = g(x).$$

Show that solution exists if and only if $g(x) = \frac{c}{1-cx}$

A Parametrization of the initial curve is $\{(s, \frac{s^2}{2}, g(s)) : s \in \mathbb{R}\}$. The Jacobian J is equal to

$$J = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} 1 & s \\ 1 & s \end{vmatrix} = 0.$$

So the problem is characteristic problem. As earlier let us consider the initial condition

$$u(0,y) = h(y), \ \ y \in \mathbb{R}$$

This space curve can be parametrized as $\{(0,s,h(s)); \ s \in \mathbb{R}\}$. Then the The characteristic equations are

$$\begin{aligned} \frac{dx}{dt} &= 1, \ \ x(s,0) = 0 \\ \frac{dy}{dt} &= x, \ \ y(s,0) = s \\ \frac{dz}{dt} &= z^2, \ \ z(s,0) = h(s) \end{aligned}$$

Solving this we get

$$t = x, \ y = \frac{x^2}{2} + s, \text{ and } z = \frac{h(s)}{1 - th(s)}$$

Inverting the variables we get

$$u(x,y) = z = \frac{h(y - \frac{x^2}{2})}{1 - xh(y - \frac{x^2}{2})}.$$

Therefore on $y = \frac{x^2}{2}$, $u(x, \frac{x^2}{2}) = \frac{h(0)}{1-xh(0)}$. Hence solution of the given problem exists if

$$g(x) = \frac{c}{1 - xc}.$$

In this case there are infinitely many solutions exists, a family of them is

$$u(x,y) = \frac{c + f(y - \frac{x^2}{2})}{1 - x(c + f(y - \frac{x^2}{2})}$$

for any f with $f(0) = 0$. $\square$

1.2.2 Quasilinear Equations

The most important equation that appears in the study of fluid flows is Burgers equation

$$uu_x + u_t = 0 \tag{2.3}$$

This is seen as an idealistic case of 1-d Navier Stokes equations with no pressure gradient

$$\rho \frac{\partial v}{dt} + \rho v \frac{\partial v}{\partial x} - \mu \frac{\partial^2 v}{\partial x^2} = 0$$

where ρ is the density, v is velocity and μ is the viscosity of fluid. When the viscosity of the fluid is assumed to be almost zero, this would lead to an equation of the form (1.1). In this case the initial value problem looks like

$$uu_x + u_y = 0,\ x, y \in \mathbb{R},\ \ u(x,0) = h(x),\ x \in \mathbb{R}$$

The characteristic equations are

$$\frac{dx}{dt} = z,\ \ x(s,0) = s$$
$$\frac{dy}{dt} = 1,\ \ y(s,0) = 0$$
$$\frac{dz}{dt} = 0,\ \ z(s,0) = h(s)$$

Solving this we get

$$y = t,\ z = h(s) \text{ and } x = h(s)t + s$$

The characteristic lines are $x = h(s)y + s$ and the speed is $\frac{dx}{dy} = h(s)$. Inverting the variables we get

$$t = y,\ s = x - zy,$$

and the solution can be defined implicitly as

$$u(x,y) = z = h(x - zy) = h(x - uy)$$

Now let γ_1 be the characteristic curve at s_1 with starting speed as $h(s_1)$ and let γ_2 be the characteristic curve at s_2 with starting speed $h(s_2)$. If these curves intersect, then

$$s_1 + h(s_1)y = s_2 + h(s_2)y$$

Therefore

$$y = t = \frac{s_1 - s_2}{h(s_2) - h(s_1)}$$

So $y > 0$ if $s_1 > s_2$ and $h(s_2) > h(s_1)$. That is if h is decreasing function, then the characteristic lines intersect at a point $y > 0$. This phenomena is called **Gradient catastrope**. Imagine this phenamena a wave moving from left with height h_1 is clashing another wave moving from right with height h_2 with different speeds. When the two waves meet there is a blowup.

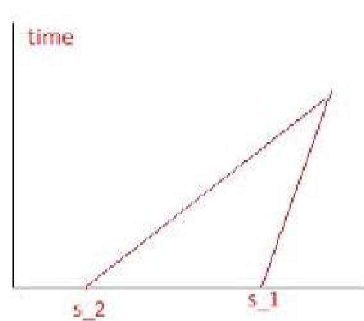


Fig. 1.2

That is if $h'(s_0) < 0$ at any s_0, then the solution does not exist "globally". On the other hand if $h'(s_0) \geq 0$, then solution exists globally. Given the original physical model, we can interpret as: If the initial velocity of the fluid flow form a non-decreasing function of position, then the fluid moves out in a smooth fashion. If the initial velocity is decreasing function, then the fluid flow undergo a "**Shock**" that correspond to collision of particles. That is the integral surface folds itself.

Example 1.5. Solve the problem

$$uu_x + yu_y = x, \;\; x, y \in \mathbb{R}, \; u(x,1) = 2x, \; x \in \mathbb{R}$$

A parametrization of Γ is $\{(s,1,2s) : \; s \in \mathbb{R}\}$. The Jacobain is

$$J = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} 2s & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

The characteristic equations are

$$\frac{dx}{dt} = z, \;\; x(s,0) = s$$
$$\frac{dy}{dt} = y, \;\; y(s,0) = 1$$
$$\frac{dz}{dt} = x, \;\; z(s,0) = 2s$$

The solution of this problem is

$$y(s,t) = c(s)e^t, \; y(s,0) = 1 \implies y = e^t$$

$$\frac{d}{dt}(x+z) = x+z, \text{ and } \frac{d}{dt}(x-z) = z-x$$

Therefore

$$x+z = 3se^t, \;\; x-z = -se^t$$

Simplifying this we get

$$x = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \; z = \frac{3}{2}se^t + \frac{1}{2}se^{-t}, y = e^t$$

Hence

$$u(x,y) = x\frac{3y^2+1}{3y^2-1}$$

We have the following important example where the initial condition is only continuous function.

Example 1.6. Find the solution of the initial value problem near the initial curve:

$$u_y + uu_x = 0, x \in \mathbb{R}, y > 0, \;\; u(x,0) = h(x) = \begin{cases} 1 & x < 0 \\ 1-x & x \in (0,1) \\ 0 & x > 1 \end{cases}$$

The characteristic lines are $y(s,t) = t$ and

$$x(s,t) = s + th(s)$$
$$= \begin{cases} s+t & s \leq 0 \\ s+t(1-s) & s \in (0,1), \\ s & s \geq 1 \end{cases}$$

For $y < 1$, the characteristic lines does not intersect. So given a point (x,y) with $y < 1$, we can draw the backward through characteristics

$$s = \begin{cases} x-y & x < y \\ \frac{x-y}{1-y} & y \leq x < 1 \\ x & x \geq 1 \end{cases}$$

The characteristic lines are show below.

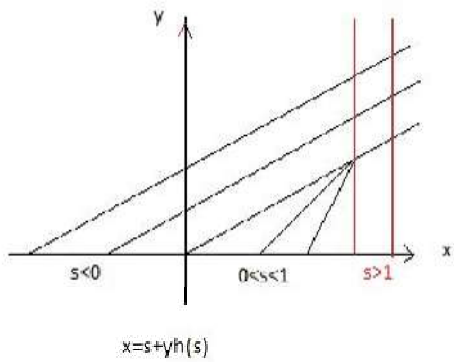


Fig. 1.3 local solution

The solution for $y < 1$ is computed as follows:

$$s = x - y,\ s \leq 1 \implies x \leq y$$

$$s = x - y + ys,\ s \in (0,1) \implies s = \frac{x-y}{1-y}$$

$$s = x, s \geq 1 \implies x \geq 1$$

Therefore,

$$u(x,y) = h(s) = \begin{cases} 1 & x < y < 1 \\ 1 - \frac{x-y}{1-y} & y \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

At the point $y = 1$ we see that u is

$$u(x,y) = \begin{cases} 1 & x < 1 \\ 0 & x > 1 \end{cases}$$

1.2.3 Propagation of discontinuities

There are situations where the initial condition has a discontinuity. In this case we cannot expect the solution to be classical (C^1) solution. One such problem is Riemann problem. To understand the propagation of the discontinuity along the characteristic curve let us consider the following problem:

Example 1.7. Consider the initial value problem

$$u_y + uu_x = 0,\ x \in \mathbb{R}, y > 0,$$
$$u(x,0) = h(x) = \begin{cases} 1 & x < 0, \\ 0 & x > 0. \end{cases}$$

The characteristic curves are

$$x = h(s)y + s = \begin{cases} y+s, & s < 0, \\ s & s > 0 \end{cases}$$

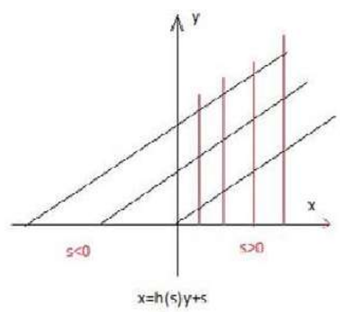


Fig. 1.4 propagation of discontinuity and shock

The characteristic lines intersect for all $x > 0$.
If we choose the initial condition as

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Then the characteristic lines are

$$x = h(s)y + s = \begin{cases} s, & s < 0, \\ y+s & s > 0 \end{cases}$$

The characteristic lines does not intersect. It is important to understand how one can define the solution in this type of situations. We can use integration by parts to define the so called generalized solutions or weak solutions.
We recall the following integration by parts formula: Let $\Omega \subset \mathbb{R}^2$ be an open set. Then

$$\int_\Omega u v_{x_1} dx = \int_{\partial\Omega} uv\eta_1 ds - \int_\Omega v u_{x_1} dx$$

where η_1 is the first component of the unit outward normal $\hat{\eta} = (\eta_1, \eta_2)$ to $\partial\Omega$ and the first integral on the right hand side is the line integral. We will study the weak solutions in the later section.

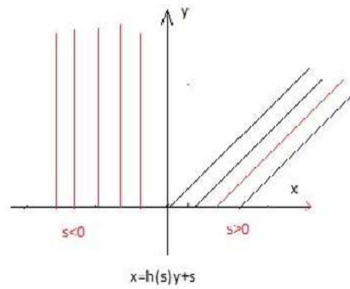


Fig. 1.5 propagation of discontinuity and fan

1.3 General solution

Theorem 1.3.1 *Suppose there exist functions ϕ and ψ such that $\phi(x,y,z)$ and $\psi(x,y,z)$ are constant along*

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}. \tag{3.4}$$

Then $F(\phi,\psi) = 0$ is the general solution of $au_x + bu_y = c$, where F is such that $F_\phi^2 + F_\psi^2 \neq 0$.

Proof. Let $x(t), y(t), z(t)$ be the solution of (3.4) and let $\phi(x(t),y(t),z(t)) = c_1$, $\psi(x(t),y(t),z(t)) = c_2$ for some constants c_1 and c_2. Then the differential of ϕ and ψ are zero. Therefore,

$$\phi_x dx + \phi_y dy + \phi_z dz = 0$$
$$\psi_x dx + \psi_y dy + \psi_z dz = 0.$$

Therefore

$$\frac{a}{\phi_y\psi_z - \psi_y\phi_z} = \frac{-b}{\phi_x\psi_z - \phi_z\psi_x} = \frac{c}{\phi_x\psi_y - \phi_y\psi_x} \tag{3.5}$$

Since $z = z(x,y)$, we have $F = F(\phi,\psi) = 0$ is a function of x,y. Therefore its total derivative is zero. That is

$$dF = F_x dx + F_y dy = 0 \implies F_x = 0, F_y = 0$$

But by chain rule,

$$F_x = F_\phi(\phi_x + \phi_z z_x) + F_\phi(\psi_x + \psi_z z_x) = 0$$

$$F_y = F_\phi(\phi_y + \phi_z z_y) + F_\phi(\psi_y + \psi_z z_y) = 0.$$

Since $(F_\phi)^2 + (F_\psi)^2 \neq 0$, the above system has non-trivial solution. That is

$$\begin{vmatrix} \phi_x + \phi_z z_x & \psi_x + \psi_z z_x \\ \phi_y + \phi_z z_y & \psi_y + \psi_z z_y \end{vmatrix} = 0.$$

Then again thank s to (3.5) this is equal to saying $az_x + bz_y = c$.

Example 1.8. Find the general solution of $uu_x + yu_y = x$.

The equations in (3.4) imply

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

i.e.,

$$xdx - zdz = 0 \implies d(x^2 - y^2) = 0$$

Therefore $\phi(x,y,z) = x^2 - z^2$.
Also

$$xdy - ydx = xy - yz = yx - zy = ydz - zdy \implies (x+z)dy - y(dx+dz) = 0$$

Therefore $d(\frac{y}{x+z}) \implies \psi = \frac{y}{x=z}$.
Therefore the general solution is

$$u^2 = x^2 + f(\frac{y}{x+u}).$$

Here we point out that the general solution does not represent all solutions.

1.4 Nonlinear Equations

In this section we will study the nonlinear equation

$$F(x,y,u,u_x,u_y) = 0$$

We introduce two more variables $p = u_x$, and $q = u_y$. So we have the five variable function

$$F(x,y,z,p,q) = 0$$

Suppose F has the quasilinear form

$$F = a(x,y,z)p + b(x,y,z)q - c(x,y,z) = 0$$

Then

$$\begin{aligned} \frac{dF}{dp} &= a = \frac{dx}{dt} \\ \frac{dF}{dq} &= b = \frac{dy}{dt} \\ p\frac{dF}{dp} + q\frac{dF}{dq} &= ap + bq = c = \frac{dz}{dt} \end{aligned}$$

Taking this as motivation, we write three equations

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{dz}{dt} = pF_p + qF_q \tag{4.6}$$

We nee equations satisfied by p and q as well. For this, we notice

$$\frac{dp}{dt} = \frac{d}{dt}u_x(x,y) = u_{xx}x'(t) + u_{xy}y'(t) = F_p p_x + F_q q_x$$

But we don't want p_x and q_x. Differentiating the equation $F = 0$ with respect to x, we get

$$F_x + F_z z_x + F_p p_x + F_q q_x = 0$$

That is

$$F_p p_x + F_q q_x = -F_x - F_z p$$

Therefore

$$\frac{dp}{dt} = -F_x - pF_z, \text{ and } \frac{dq}{dt} = -F_y - F_z q \tag{4.7}$$

These five equations in (3.5) and (4.6) form characteristic strip. The initial condition $u(\gamma) = h$ gives

$$x(s,0) = f(s),\ y(s,0) = g(s),\ z(s,0) = h(s)$$

The initial values for of p and q are obtained as follows:

$$u(f(s),g(s)) = h(s) \implies h'(s) = u_x f'(s) + u_y g'(s) = pf' + qg'$$

Therefore p_0 and q_0 must satisfy "strip condition"

$$p_0 f'(s) + q_0 g'(s) = h'(s) \tag{4.8}$$

But such p_0 and q_0 may not be few. There could be infinitely many. Also

$$F(f(s),g(s),h(s),p_0(s),q_0(s)) = 0 \tag{4.9}$$

be satisfied on the initial curve. So p_0, q_0 are such that (4.8) and (4.9) holds. The solution of these characteristic equations determine a characteristic strip similar to the curves in the semilinear case. These characteristic strips form a smooth surface which is the integral surface. This is like scales on the surface of fish forming a smooth surface on the body of fish.

Example 1.9. Solve the initial value problem

$$u_x u_y = u,\ x,y \in \mathbb{R},\ u(0,y) = y^2.$$

$F(p,q) = pq - z$ and parametrization of initial curve is $\{(0,s,s^2) : s \in \mathbb{R}\}$. Therefore $f(s) = 0, g(s) = s, h(s) = s^2$. The initial values for p and q satisfy

$$p_0 0 + q_0 1 = 2s,\ \ p_0 q_0 - h = 0 \implies p_0 2s - s^2 = 0$$

Therefore $p_0 = \frac{s}{2}$. The characteristic strip satisfy

$$\begin{aligned}
\frac{dx}{dt} &= F_p = q, \quad x(s,0) = 0 \\
\frac{dy}{dt} &= F_q = p, \quad y(s,0) = s \\
\frac{dz}{dt} &= pF_p + qF_q = pq + pq = 2pq = 2z, \quad z(s,0) = s^2 \\
\frac{dp}{dt} &= -F_x - F_z p = -p, \quad p(s,0) = \frac{s}{2} \\
\frac{dq}{dt} &= q, \quad q(s,0) = 2s
\end{aligned}$$

Therefore $p(s,t) = \frac{1}{2}se^t,\ q(s,t) = 2se^t$ and $z(s,t) = s^2 e^{2t}$.

$$\frac{dx}{dt} = 2se^t \implies x = 2se^t - 2s$$

$$\frac{dy}{dt} = p = \frac{1}{2}se^t + \frac{s}{2}$$

Therefore

$$\frac{x}{2} + 2y = 2se^t$$

squaring this we get

$$(\frac{x}{4} + y)^2 = s^2e^{2t} = z$$

Therefore $u(x,y) = \left(\frac{x}{4} + y\right)^2$.

1.4.1 Complete integral and Singular solutions

Let $A \subset \mathbb{R}^2$ be an open set which is parameter set. Let

$$(D_a u, D^2_{xa} u) = \begin{bmatrix} u_{a_1} & u_{xa_1} & u_{ya_1} \\ u_{a_2} & u_{xa_2} & u_{ya_2} \end{bmatrix}$$

Definition 1.4.1 *A C^2 function $u = u(x,a)$ is said to be a complete integral in $\Omega \times A$ of the equation*

$$F(x,y,u,u_x,u_y) = 0 \text{ in } \Omega \tag{4.10}$$

if $u(x,a)$ solves (4.10) *and rank of* $(D_a u, D^2_{xa} u) = 2$.

Example 1.10. Find a complete integral of $u_x u_y = u$

From the given equation $F = pq - z$ and the characteristic equations are

$$\frac{dx}{dt} = q,\ \frac{dy}{dt} = p,\ \frac{dz}{dt} = 2z,\ \frac{dp}{dt} = p\ \frac{dq}{dt} = q.$$

On integrating we get

$$p = c_1e^t, q = c_2e^t, z = c_1c_2e^{2t} + c_3, x = c_2e^t + b, \text{ and } y = c_1e^t + a$$

Therefore

$$u(x,y) = z = (x-b)(y-a) + c_3$$

for $u(x,y)$ to be a solution we need $c_3 = 0$. Then we get

$$u(x,y) = xy + ab + (a,b) \cdot (x,y)$$

Easy to check that

$$(D_a u, D^2_{xa} u) = \begin{bmatrix} b+x & 1 & 0 \\ a+y & 0 & 1 \end{bmatrix}$$

whose rank is 2. Therefore $u(x,y)$ is a complete integral.
There can be more than one complete integral as we see below

$$p = c_1e^t, q = c_2e^t \implies \frac{p}{q} = a \text{ and } pq = z$$

This implies

$$p = \pm\sqrt{az} \;\; q = \pm\sqrt{\frac{z}{a}}$$

From the strip condition, we get

$$\begin{aligned}\frac{dz}{dt} &= p\frac{dx}{dt} + q\frac{dy}{dt} \\ &= \pm\sqrt{az}\frac{dx}{dt} \pm \sqrt{\frac{z}{a}}\frac{dy}{dt}\end{aligned}$$

Integrating this

$$2\sqrt{z} = \pm(\sqrt{a}x + \frac{y}{\sqrt{a}}) + c_3$$

Therefore

$$u(x,y) = b + \frac{1}{2}(\sqrt{a}x + \frac{y}{\sqrt{a}})^2$$

is a complete integral if $a > 0$. (check!).

In the above example, in both complete integral, No choice of a and b will give trivial solution $u = 0$. But $u = 0$ is also a solution. This motivates us to study

1.4.2 Singular solutions

In this section we study how to build more complicated solutions for nonlinear first order PDEs. We will construct these new solutions as envelopes of complete integrals.

Definition 1.4.2 *Suppose $u(x,y,a_1,a_2)$ be a C^1 function on $U \times A$ where U is open subset of $\mathbb{R}^2$ and A be an open subset of $\mathbb{R}^2$. If the equations*

$$\frac{\partial u}{\partial a_i}(x,y,a_1,a_2) = 0, \; x \in U, a \in A, \; i = 1,2...n$$

can be solved for the parameter a and has solution $a_1 = \phi(x,y)$, $a_2 = \psi(x,y)$. That is

$$\frac{\partial u}{\partial a_i}(x,y,\phi(x,y),\psi(x,y)) = 0, \; i = 1,2,...,n.$$

Then we call the function $v(x) = u(x,y,\phi(x),\psi(x))$, the envelope of the functions $\{u(.,a)\}_{a\in A}$.

Theorem 1.4.1 *Singular solutions:*
Suppose for each $(a_1,a_2) \in A$ as above that $u = u(.,a_1,a_2)$ solves the partial differential equation (4.10). *Assume further that the envelope v, defined by*

$$\frac{\partial u}{\partial a_i}(x,y,a_1,a_2) = 0, \; x \in U, a \in A, \; i = 1,2...n$$

$$v(x,y) = u(x,y,\phi(x,y),\psi(x,y))$$

exists and is a C^1 function. Then v also solves (4.10).

Proof. We have $v(x,y) = u(x,y,\phi(x,y),\psi(x,y))$ and so

$$v_x = u_x(x,y,\phi,\psi) + u_{a_1}(x,y,\phi,\psi)\phi_x + u_{a_2}(x,y,\phi,\psi)\phi_x = u_x(x,y,\phi,\psi)$$
$$v_y = u_y(x,y,\phi,\psi) + u_{a_1}(x,y,\phi,\psi)\phi_y + u_{a_2}(x,y,\phi,\psi)\phi_y = u_x(x,y,\phi,\psi)$$

Hence for each $x \in U$,

$$F(x,y,u,u_x,u_y) = F(x,y,u(x,y,\phi,\psi),u_x(x,y,\phi,\psi),u_y(x,y,\phi,\psi)) = 0$$

Example 1.11. Find complete integral and singular solution of

$$u_x^2 + u_y^2 = 1 + 2u$$

Here $F = p^2 + q^2 - 1 - 2z$ and the characteristic equations are

$$\frac{dx}{dt} = 2p,\ \frac{dy}{dt} = 2q,\ \frac{dp}{dt} = 2p,\ \frac{dq}{dt} = 2q$$

solving this we get $\frac{p}{q} = a$ and $p^2 + q^2 - 1 - 2z = 0$, This implies

$$(1+a^2)q^2 = 1 + 2z$$

Therefore

$$p = \pm a\sqrt{\frac{1+2z}{1+a^2}},\ q = \pm\frac{1+2z}{1+a^2}$$

Now from the strip condition

$$\begin{aligned}\frac{dz}{dt} &= p\frac{dx}{dt} + q\frac{dy}{dt}\\ &= \pm\sqrt{\frac{1+2z}{1+a^2}}(adx+dy)\end{aligned}$$

Integrating

$$\sqrt{1+2z} = \pm\frac{ax+y}{\sqrt{1+a^2}} \pm b$$

Therefore

$$u(x,y,a,b) = \frac{1}{2}\left(\frac{ax+y}{\sqrt{1+a^2}} + b\right)^2 - \frac{1}{2}$$

is the complete integral. Now solving

$$u_a = 0,\ \ u_b = 0 \implies \frac{ax+y}{\sqrt{1+a^2}} + b = 0.$$

Therefore singular solution is $u = -\frac{1}{2}$.

To generate more solutions from the complete integral, we vary the above construction. Choose an open set $A' \subset \mathbb{R}$ and any C^1 function $h : A' \to \mathbb{R}$, so that the graph of h lies within A.

Definition 1.4.3 *The general integral (depending on h) is the envelope $v(x)$ of the functions*

$$u(x,y,a_1) = u(x,y,a_1,h(a_1)),\ (x,y) \in U,\ a \in A'$$

provided this envelope exists and is C^1.

Example 1.12. Find a general integral of $u_x u_y = u$.

Earlier we found a complete integral as

$$u(x,y,a,b) = (x-a)(y-b) = xy + ab - ay - bx$$

Let $h : \mathbb{R} \to \mathbb{R}$ be $h(a) = a$. Then

$$u(x,y,a,a) = xy + a^2 - ay - ax$$

$$u_a = 2a - y - x = 0 \implies a = \phi(x,y) = \frac{x+y}{2}$$

Therefore

$$u(x,y,\phi(x,y)) = xy + \frac{(x+y)^2}{4} - \frac{x+y}{2}y - \frac{x+y}{2}x = -\frac{1}{4}(x-y)^2.$$

1.5 Weak solutions

In this section we will study the weak solutions of the Initial value problem

$$u_t + (f(u))_x = 0,\ t > 0,\ x \in \mathbb{R}, \tag{5.11}$$

$$u(x,0) = h(x),\ x \in \mathbb{R}. \tag{5.12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is a smooth function with compact support. Multiplying (5.11) with $v(x,t)$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty u_t v \, dxdt + \int_0^\infty \int_{-\infty}^\infty (f(u))_x v \, dxdt \\ &= -\int_0^\infty \int_{-\infty}^\infty u v_t \, dxdt - \int_0^\infty \int_{-\infty}^\infty (f(u)) v_x \, dxdt + \int_{-\infty}^\infty u(x,0) v(x,0) \, dx \end{aligned}$$

Motivated by this, we define

Definition 1.5.1 *We call any bounded measurable function* $u(x,t)$ *as a weak solution if*

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + (f(u)) v_x) \, dxdt = \int_{-\infty}^\infty u(x,0) v(x,0) \, dx$$

for all $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$.

It is easy to see that all classical solutions are weak solutions. But not all weak solutions are classical solutions. If u is a weak solution, then it need not be even continuous. Suppose $u(x,t)$ is a weak solution of (5.11) such that $u(x,t)$ is discontinuous across a curve $x = x(t)$, but $u(x,t)$ is smooth on either side of $x(t)$. Let $u^-(x,t)$ be the limit of u approaching (x,t) from the left and let $u^+(x,t)$ be the limit of u approaching $x(t)$ from right. We claim that such a curve $x(t)$ cannot be arbitrary.

Theorem 1.5.1 *(Rankine-Hugoniot condition)*
If $u(x,t)$ *is a weak solution of* (5.11) *above such that* $u(x,t)$ *is discontinuous across the curve* $x = x(t)$ *but* u *is smooth on either side of curve* D^- *and* D^+*. Then* $u(x,t)$ *must satisfy the condition*

$$x'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+}$$

where u^- is the limit of u from D^- and u^+ is limit of u from D^+.

Proof. Let $D^- = \{(x,t) : 0 < t < \infty, -\infty < x < x(t)\}$ and $D^+ - = \{(x,t) : 0 < t < \infty, x(t) < x < \infty\}$. Then if $u(x,0) = 0$, we have by divergence theorem on D^- we get

$$\iint_{D^-} [uv_t + f(u)v_x]\, dxdt = -\iint_{D^-} [u_t + (f(u))_x]\, vdxdt + \int_{x=x(t)} \left(u^- vn_2 + f(u^-)vn_1\right) ds \tag{5.13}$$

where $n = (n_1, n_2)$ is the outward unit normal to D^-. Similarly,

$$\iint_{D^+} [uv_t + f(u)v_x]\, dxdt = -\iint_{D^+} [u_t + (f(u))_x]\, vdxdt - \int_{x=x(t)} \left(u^+ vn_2 + f(u^+)vn_1\right) ds \tag{5.14}$$

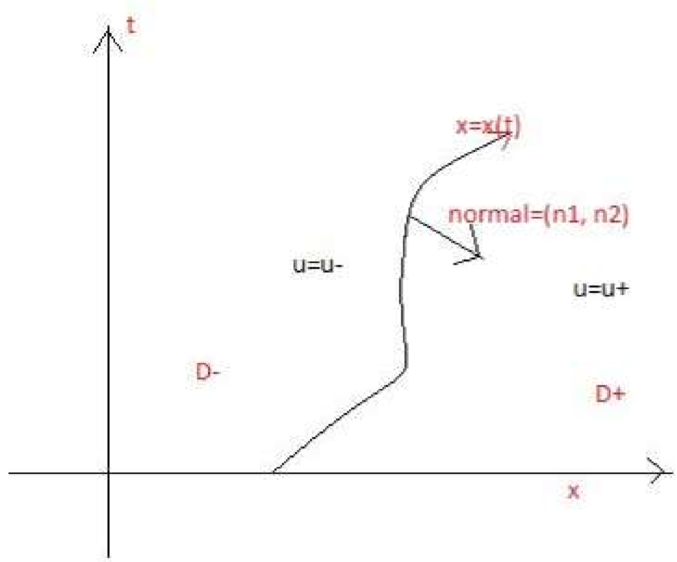


Fig. 1.6 weak solution

By assumption $u_t + (f(u))_x = 0$ in D^- and D^+. Therefore from (5.11) and (5.13) we get

$$\int_{x=x(t)} \left(u^- vn_2 + f(u^-)vn_1\right) ds = \int_{x=x(t)} \left(u^+ vn_2 + f(u^+)vn_1\right) ds$$

holds for all $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$. Therefore

$$u^- n_2 + f(u^-)n_1 = u^+ n_2 + f(u^+)n_1$$

This implies

$$-\frac{n_2}{n_1} = \frac{f(u^-) - f(u^+)}{u^- - u^+}$$

On the curve $x = x(t)$,

$$\frac{dx}{dt} = \frac{1}{x'(t)} = -\frac{n_2}{n_1}.$$

Hence the result.

Recall the example in 1.6. We defined the classical solution for $t < 1$(take $y = t$). Now beyond the line $y = 1$, we assume that the solution is smooth on either side of a curve $x = x(t)$. We note that $u^{-}0, u^{=}1$. BY Rankine-Hugoniot condition,

$$x'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{1}{2}$$

So $x = x(t) = \frac{t}{2}, x(1) = 1$. Therefore the equation of shock is $x = \frac{t}{2} + 1$. Hence

$$u(x,t) = \begin{cases} 1 & x < \frac{1+t}{2} \\ 0 & x > \frac{1+t}{2}. \end{cases}$$

We can even define weak solution in case the initial condition has discontinuity like in the example 1.7. In the first case

$$x'(t) = \frac{1}{2}, x(0) = 0 \implies x = \frac{t}{2}$$

The solution is

$$u(x,t) = \begin{cases} 1 & x < \frac{t}{2} \\ 0 & x > \frac{t}{2} \end{cases}$$

In the second case, the wave on the right moves faster and the wave on the left moves slower. So there is no shock. In this case one can define continuous solution which are called rarefaction wave solution like

$$u_1(x,t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 \leq x \leq t \\ 1 & x \geq t \end{cases}$$

But even in this case one can introduce shock and define shock wave solution as

$$u_2(x,t) = \begin{cases} 0 & x < \frac{t}{2} \\ 1 & x > \frac{t}{2} \end{cases}$$

The solution u_2 is NOT physically feasible solution. But the solution in u_1 is more realistic which is known as entropy solution.

1.6 Problems

1. Solve the given IVP and determine the values of x and y for which it exists;

   a. $xu_x + u_y = y, \ u(x,0) = x^2$.
   b. $u_x - 2u_y = u, \ u(0,y) = y$.
   c. $y^{-1}u_x + u_y = u^2, \ u(x,1) = x^2$

2. Solve the given IVP and determine the values of x,y and z for which it exists:

a. $xu_x + yu_y + u_z = u, \; u(x,y,0) = h(x,y)$
b. $u_x + u_y + zu_z = u^3, \; u(x,y,1) = h(x,y)$

3. Solve the given IVP and determine the values of x and y for which it exists;

$$\text{(a) } u_x + u^2 u_y = 1, \; u(x,0) = 1 \quad \text{(b) } u_x + \sqrt{u} u_y = 0, \; u(x,0) = x^2 + 1$$

4. Find a general solution:

$$\text{(a) } (x+u)u_x + (y+u)u_y = 0 \quad \text{(b) } (x^2 + 3y^2 + 3u^2)u_x - 2xyu_y + 2xu = 0$$

5. Consider the equation $u_x + u_y = \sqrt{u}$. Derive the general solution $u(x,y) = (x + f(x-y))^2/4$. Observe that the trivial solution $u(x,y) \equiv 0$ is not covered by the general solution.
6. Solve the IVP $a(u)u_x + u_y = 0$ with $u(x,0) = h(x)$, and show that solution become singular for some $y > 0$ unless $a(h(s))$ is a nondecreasing function of s.
7. Solve $u_x^2 + yu_x - u = 0$ with initial condition $u(x,1) = \frac{x^2}{4} + 1$.
8. Solve $u = xu_x + yu_y + (u_x^2 + u_y^2)/2$ with initial condition $u(x,0) = (1 - x^2/2)$
9. Consider $u = u_x^2 + u_y^2$ with the initial condition $u(x,0) = ax^2$. For what positive constants a is there a solution? Is it unique? Find all solutions.
10. Show that family of spheres S_a given by $(x-a)^2 + y^2 + z^2 = 1$ has as its envelope ζ the unit cylinder $y^2 + z^2 = 1$.
11. Consider $c^2(u_x^2 + u_y^2) = 1$, where $c = c(x,y)$. Derive the characteristic equations. In the special case $c = |x|$ with initial condition $u(x,0) = 0$, find the solution to be

$$u(x,y) = -\log \frac{\sqrt{x^2+y^2} + y}{x}, \quad \text{for } x > 0$$

12. Consider the eikonal equation in three dimensions $u_x^2 + u_y^2 + u_z^2 = 1$

    a. Solve the initial value problem with $u = k =$ constant on the plane $\alpha x + \beta y + z = 0$
    b. Find a complete integral.

13. Find complete integral of
$(a)\; uu_xu_y = u_x + u_y \; (b)\; u_x^2 + u_y^2 = xu$
14. Given the complete integral, find the singular solution of

    a. $u_x u_y = z, \qquad u = xy + ab + (ab)(xy)$
    b. $u_x^2 + u_y^2 = 1 \qquad u = (xy)(\cos a . \sin a) + b$

15. Consider $uu_x + u_y = 0$ with the initial condition

$$u(x,0) = h(x) = \begin{cases} 0 & x < 0 \\ u_0(x-1) & x > 0, \end{cases}$$

where $u_0 > 0$. There is a weak solution $u(x,y)$ that has a jump discontinuity along a curve $x = \xi(y)$. Find this curve and describe the weak solution.
16. Find a simple wave equation $u(x,y) = u(\frac{x}{y})$ for $(G(u))_x + u_y = 0$, when $G(u) = u^4/4$. Use this to define a continuous weak solution of the problem for $y > 0$ that satisfies

$$h(x) = \begin{cases} 0 & x > 0 \\ -1 & x < 0, \end{cases}$$

17. Consider $uu_x + u_y = 0$ with

$$h(x) = \begin{cases} 0 & x > 0 \\ u_0 & x < 0, \end{cases}$$

where $u_0 < 0$. In addition to the rarefaction solution described in the text, show that there is a weak solution with a shock along $x = u_0 y/2$.

18. A reasonable model for low-density traffic is $\frac{dq}{d\rho}\rho_x + \rho_t = 0$ with $\frac{dq}{d\rho} = c$, where c is a constant.

   a. Show that ρ is constant along the (characteristic) curves $x = ct + x_0$.
   b. If car is alone on the highway, what does $\rho(x,0)$ look like? What does $\rho(x,t)$ look like ?
   c. Explain why c represents the free speed of highway.

19. If $\rho_{\max}$ denotes the maximum density of cars on a highway(i.e. under bumper-to-bumper conditions), then a reasonable relation between q and ρ is given by $G(\rho) = c\rho(1 - \frac{\rho}{\rho_{\max}})$ where the constant c is the free speed of the highway (cf. Exercise 16). Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_{\max} & x < 0 \\ \rho_{\max} & x > 0, \end{cases}$$

Find the shock curve and describe the weak solution. Interpret your result for the traffic flow.

20. Using $G(\rho)$ as in Exercise above, describe the traffic flow after a long red light turns green at $t = 0$; that is , the initial density is

$$\rho(x,0) = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0, \end{cases}$$

In particular, find the density at the green light, $\rho(0,t)$, while the light remains green.

21. Consider $uu_x + u_t = 0$ with the initial condition

$$u(x,0) = h(x) = \begin{cases} 2 & x < 0 \\ 1 & x \in (0,1) \\ 0 & x > 1, \end{cases}$$

Describe the weak solution. Show that there is only one shock as t is large.