Chapter 5

Fourier Methods

5.1 Boundary Value Problems

The first and foremost concept to recall here is the Fredholm alternative for the system of linear equations. Let A be an $n \times n$ matrix and consider the systme of equation $Ax = b$. We know the following:

1. The system has a unique solution (in case $det(A) \neq 0$).
2. The system has no solution or the system is consistent if and only if $\langle b, x\rangle = 0$ for all $x \in \{y : Ay = 0\}$.

But while solving boundary value problems, we have to find equivalent way of determining $det(A) \neq 0$. Let us assume that the eigenvectors form a basis of $\mathbb{R}^n$. Let $\mathscr{B} = \{e_1, e_2, e_3, ...e_n\}$ be an orthonormal basis of eigenvectors of A. That is $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and $\langle e_i, e_i\rangle = 1$.

Now let $b \in \mathbb{R}^n$. Then $b = \sum_{i=1}^{n} b_i e_i$, where $b_i = \langle b, e_i\rangle$. Since $\mathscr{B}$ is a basis of $\mathbb{R}^n$, we assume that the solution to be

$$x = x_1e_1 + x_2e_2 + ... + x_ne_n$$

So it is enough to solve for $x_i's$. Now substituting this in the equation $Ax = b$ we get

$$\sum_{i=1}^{n} \lambda_i x_i e_i = A\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} b_i ei \implies \sum_{i=1}^{n} x_i A e_i = \sum_{i=1}^{n} b_i e_i$$

Since $e_1, e_2, ...e_n$ are Linearly independent, we get

$$\lambda_i x_i = b_i, \;\; i = 1, 2, ...n. \tag{1.1}$$

From the equation (1.1), we get the following assertions:

1. If zero is not an eigenvalue of A then the problem $Ax = b$ has unique solution $x = \sum_{i=1}^{n} \frac{b_i}{\lambda_i} e_i$. That is, if A is non-singular, then we have unique solution.
2. If $\lambda_m = 0$ for some m. Then solutions exists if $b_i = 0$, in which case x_m is arbitrary. Since $\mathscr{B}$ is orthonormal, $b_m = \langle b, e_m\rangle = 0$. Also e_m satisfies $Ae_m = 0$. That is solutions exists if b is orthogonal to every solution of $Ax = 0$.
3. On the other hand, if b is othogonal to every solution of $Ax = 0$. Then $\lambda_m = 0$ is an eigenvalue of A and $b_m = 0$. Hence x_m is arbitrary in the equation (1.1) and infinitely many solutions exist.
4. It is also clear that the problem $Ax = b$ has NO SOLUTION if $\lambda_m = 0$ and $\langle b, e_m\rangle \neq 0$.

The above method tells us that once we know the eigenvalues and eigenvectors forms basis of $\mathbb{R}^n$, then it is easy to determine if the system has infinitely many solutions or there is no solution. We will show that we can generalize the above approach to differential equations to solve a non-homogeneous boundary value problem.
Now we consider the boundary value problem

$$L(y) = f(x),\ y(0) = 0,\ y(1) = 0$$

where $L(y) = -y''$ or a self adjoint positive definit operator $-(py')' + qy$ for some continuous functions $p(x) > 0$ and $q(x)$. One can prove the same assertions as in (1)-(4) above by constructing Green's function. Basically one can prove that there exists unique Green's function if $L(y) = 0, y(0) = 0, y(1) = 0$ has only trivial solution. In this case the unique solution is given by

$$u(x) = \int_0^1 G(x,y)f(y)dy.$$

In case of $\int_0^1 f(x)\phi(x)dx = 0$ for all ϕ satisfying $L(y) = 0,\ y(0) = 0, y(1) = 0$, one can show that there are infinitely many Green's functions. Also one can show using the Sturm's oscillation theory one can show that there are infinitely many eigenvalues $\lambda_n \to \infty$ and $\{\phi_n\}$ the sequence of eigen functions which form a complete set for $C([0,1])$. In this case again one can replicate the idea of writing the solution in the form of series $\sum c_n\phi_n$ and find the constants c_n so that u satisfies $L(u) = f$. This is known as the Fourier series method. This way we can again see the Fredholm alternative theorems.

5.2 Fourier Series

Let $C([0,1])$ be the set of all continuous functions on $[0,1]$ equipped with inner product and norm

$$(f,g) = \int_0^1 f(x)g(x)dx \text{ and } \|f\|^2 = (f,f).$$

Let $k(x,\xi)$ be a continuous real valued function on $[0,1]\times[0,1]$ such that $k(x,\xi) = k(\xi,x)$ and let $K : C([0,1]) \to C([0,1])$ be defined as

$$K(u)(x) = \int_0^1 k(x,\xi)u(\xi)d\xi$$

Then it is easy to check that K is bounded linear operator. Also it is not difficult to check that for $u,v \in C([0,1])$ with zero boundary values, one has

$$(Ku,v) = (u,Kv)$$

Theorem 5.2.1 *The set $\{Ku, \|u\| \le 1\}$ is compact in $C([0,1])$.*

Proof. Since $k(x,\xi)$ is continuous on closed bounded set, we have $|k(x,\xi)| \le M$ for some M and $k(x,\xi)$ is uniformly continuous on $[0,1]\times[0,1]$. Therefore for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x-y| \le \delta \implies |k(x,\xi) - k(y,\xi)| < \varepsilon$$

Hence, for $|x-y|<\delta$, we have

$$\begin{aligned}|Ku(x)-Ku(y)| &= \left|\int_0^1 [k(x,\xi)-k(y,\xi)]u(\xi)d\xi\right| \\ &\le \varepsilon \int_0^1 |u(\xi)|d\xi \le \varepsilon\|u\| \le \varepsilon,\ \forall u \in B_1(0)=\{u:\|u\|\le 1\}\end{aligned}$$

Also

$$|Ku(x)| \le \int_0^1 |k(x,\xi)|u(\xi)d\xi \le M\int_0^1 |u(\xi)|d\xi \le M,\ \forall u \in B_1(0)$$

Therefore, by Ascoli-Arzela theorem, we see that $K(B_1)$ is compact. Now the operator norm of K is

$$\|K\| = \sup_{\|u\|=1} \|Ku\|.$$

Then we have

Theorem 5.2.2

$$\|K\| = \sup_{\|u\|=1} |(Ku,u)|$$

Proof. Let $\eta = \sup_{\|u\|=1}|(Ku,u)|$. Then by Cauchy-Scwartz inequality, for u with $\|u\|=1$, we get

$$|(Ku,u)| \le \|Ku\|\|u\| \le \|Ku\| \le \|K\|.$$

therefore $\eta \le \|K\|$. Also using the relations

$$\begin{aligned}&(K(u+v),u+v) = (Ku,u)+(Ku,v)+(Kv,u)+(Kv,v)\\ &(K(u+v),u+v) \le \eta\|u+v\|^2\\ &(K(u-v),u-v) \ge -\eta\|u-v\|^2\end{aligned}$$

Using these relations, we get

$$\begin{aligned}4(Ku,v) &= (K(u+v),u+v)-(K(u-v),u-v)\\ &\le \eta[\|u+v\|^2+\|u-v\|^2]\\ &= 2\eta\left(\|u\|^2+\|v\|^2\right)\end{aligned}$$

Now taking $v=\frac{Ku}{\|Ku\|}$ for $\|u\|=1$, we get

$$4(Ku,v) = 4(Ku,\frac{Ku}{\|Ku\|}) = 4\|Ku\|$$

and

$$2\eta\left(\|u\|^2+\|v\|^2\right) = 2\eta(1+1) = 4\eta$$

which implies $\|Ku\| \le \eta$. Hence $\|Ku\| = \eta$. □

Theorem 5.2.3 *Either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

Proof. Suppose $\mu_1 = \|K\| = \sup_{\|u\|=1}(Ku,u)$. Then we can assume that there exists a sequence $\{u_n\}$ such that $\|u_n\|=1$ and $(Ku_n,u_n)\to\mu_1$. From the above theorem we know that $\{Ku_n\}$ is compact in $C([0,1])$.

Then there exists a subsequence $\{u_n\}$ (we can still denote with same notation!) such that Ku_n converges uniformly to a function ϕ_1. This uniform convergence also implies $\|Ku_n - \phi_1\| \to 0$. Also, $\|Ku_n\| \to \|\phi_1\|$. Moreover

$$\begin{aligned} 0 \leq \|Ku_n - \mu_1 u_n\|^2 &= \|Ku_n\|^2 + \mu_1^2 \|u_n\|^2 - 2\mu_1 (Ku_n, u_n) \\ &= \|Ku_n\|^2 + \mu_1^2 - 2\mu_1 (Ku_n, u_n) \to \|\phi_1\|^2 - \mu_1^2. \end{aligned}$$

Therefore $\|\phi_1\|^2 \geq \mu_1^2 > 0$. this implies $\phi_1 \not\equiv 0$. Also since $\|Ku_n\| \leq \mu_1$, we get from the above equations

$$0 \leq \|Ku_n - \mu_1 u_n\|^2 \leq \mu_1^2 + \mu_1^2 - 2\mu_1^2 = 0 \text{ as } n \to \infty.$$

Therefore, $\|Ku_n - \mu_1 u_m\| \to 0$. Now we use the 3ε argument to show that ϕ is an eigen function.

$$K\phi_1 - \mu_1 \phi_1 = K\phi_1 - K(Ku_n) + K(Ku_n) - \mu_1 Ku_n + \mu_1 Ku_n - \mu_1 \phi_1.$$

This implies

$$\begin{aligned} 0 \leq \|K\phi_1 - \mu_1 \phi_1\| &\leq \|K(\phi_1 - Ku_n)\| + \|K(Ku_n - \mu_1 u_n)\| + \mu_1 \|Ku_n - \phi_1\| \\ &\leq \|K\| \|Ku_n - \phi_1\| + \|K\| \|Ku_n - \mu_1 u_n\| + \mu_1 \|Ku_n - \phi_1\| \to 0. \end{aligned}$$

Hence the proof. □

Once we obtained the eigenvalue μ_1 and the corresponding eigen function ϕ_1, we define the new operator

$$k_1(x,\xi) = k(x,\xi) - \mu_1 \phi_1(x) \phi_1(\xi)$$

and the operator $K_1 : C([0,1]) \to C([0,1])$ as

$$K_1(u)(x) = \int_0^1 k_1(x,\xi) u(\xi) d\xi$$

So if $K_1 \neq 0$, then it is not difficult to see that K_1 is symmetric, self adjoint and compact and one can proceed to get μ_2 and ϕ_2 satisfying

$$\mu_2 = \sup_{\|u\|=1} (K_1 u, u).$$

Then $\mu_2 \leq \mu_1$ and $K_1 \phi_2 = \mu_2 \phi_2$. Moreover,

1. if $(u, \phi_1) = 0$ then $K_1 u = Ku$
2. $K_1 \phi_1 = K\phi_1 - \mu_1 \phi_1 (\phi_1, \phi_1) = \mu_1 \phi_1 - \mu_1 \phi_1 = 0$
3. $(K_1 u, \phi_1) = (Ku, \phi_1) - \mu_1 (\phi_1, \phi_1)(u, \phi_1) = (Ku, \phi_1 - \mu_1 (u, \phi_1) = 0) \; \forall u$
4. $(\phi_2, \phi_1) = \mu_2^{-1} (K_1 \phi_2, \phi_1) = \mu_2^{-1} (\phi_2, K_1 \phi_1) = 0$
5. $K\phi_2 = K_1 \phi_2 = \mu_2 \phi_2$.

Therefore μ_2 is an eigenvalue of K. Proceeding this way, we can define

$$k_m(x,\xi) = k(x,\xi) - \sum_{i=1}^{m} \mu_i \phi_i(x) \phi_i(\xi)$$

and

$$K_m u(x) = \int_0^1 k_m(x,\xi) u(\xi) d\xi$$

This process terminates only if $\|K_m\| = 0$. In this case for any $u \in C([0,1])$

$$0 = K_m u = Kf - \sum_{i=1}^{m} \mu_i \phi_i(x)(u, \phi_i)$$

that is

$$Ku = \sum_{i=1}^{m} \mu_i (u, \phi_i)\phi_i(x).$$

If $\|K_m\| > 0$ for all m, then there exists infinite sequence of eigenvalues $\{\mu_i\}$ and eigen functions $\{\phi_i\}$.

Theorem 5.2.4 *μ_i's are either finite or $\mu_i \to 0$ as $i \to \infty$.*

Proof. Suppose μ_i are infinite sequence. suppose it is bounded from below, say $|\mu_i| > c$ for some $c > 0$. Then

$$\|K\phi_n - K\phi_m\| = \mu_n^2 + \mu_m^2 > 2c^2$$

That means we have a bounded sequence $\{\phi_i\}$ and compact K such that $K\phi_i$ is not compact. Contradiction. □

Definition 5.2.1 *For any function $u \in C([0,1])$ the Fourier series of the function u is defined as*

$$\sum_{i=1}^{\infty} (u, \phi_i)\phi_i.$$

The constants (u, ϕ_i) are called Fourier coefficients of u.

Theorem 5.2.5 *Bessel's inequality: Let $f \in C([0,1])$ and let $\{\phi_i\}$ be any orthonormal sequence of functions in $C([0,1])$. Then Fourier coefficients of f satisfies*

$$\sum_{i=1}^{m} |(f, \phi_i)|^2 \le \|f\|^2, \ \forall\, m.$$

Proof. Proof follows from the fact that $\|f - \sum_{i=1}^{m} (f, \phi_i)\phi_i\| \ge 0$. Using the orthogonality of $\phi_i's$

$$0 \le \|f - \sum_{i=1}^{m} (f, \phi_i)\phi_i\|^2 = \|f\|^2 - \sum_{i=1}^{m} |(f, \phi)|^2 \|\phi_i\|^2. \qquad \square$$

This means the sequence $s_m = \sum_{i=1}^{m} |(f, \phi_i)|^2$ is increasing and bounded sequence and hence coverges to $\sum |(f, \phi)|^2$.

Theorem 5.2.6 *For any $u \in C([0,1])$, the Fourier series of Ku converges uniformly to Ku on $[0,1]$.*

Proof. Let $g_m = u - \sum_{i=1}^{m} (u, \phi_i)\phi_i$. Then $(g_m, \phi_i) = 0$ for $i = 1, 2, ...m$. Also $\|g_m\| \le \|u\|$. Therefore,

$$\|Kg_m\| \le \mu_{m+1}\|g_m\| \le |\mu_{m+1}\|u\| \to 0, \text{ as } m \to \infty.$$

That is, in the L^2 norm we have $Kg_m \to 0$. This means

$$Ku = \sum_{i=1}^{\infty} (u, \phi_i)K\phi_i = \sum_{i=1}^{\infty} \mu_i (u, \phi_i)\phi_i = \sum_{i=1}^{\infty} (u, K\phi_i)\phi_i = \sum_{i=1}^{\infty} (Ku, \phi_i)\phi_i.$$

To prove the uniform convergence, note that for any $q > p$,

$$\sum_{i=p}^{q} \mu_i (u,\phi)\phi_i = K\left(\sum_{i=p}^{q} (u,\phi_i)\phi_i\right)$$

Now using the fact that $|Ku(x)| \leq M\|u\|$, we get

$$\left|\sum_{i=p}^{q} \mu_i (u,\phi_i)\phi_i\right| \leq \left|K\left(\sum_{i=p}^{q} (u,\phi_i)\phi_i\right)\right| \leq M\sum_{i=p}^{q} |(u,\phi_i)|^2 \to 0 \text{ as } p,q \to \infty.$$

by the Bessel's inequality. □

As a consequence we have

Remark 5.2.1 *The reslts will hold true if we replace the inner product by a weighted inner product. Let the weight function $w(x) \in C([0,1])$ and $w(x) > 0$ in $[0,1]$. Then*

$$(u,v)_w = \int_0^1 u(x)v(x)w(x)dx$$

5.2.1 Boundary Value problems

We consider the self-adjoint operator $\mathscr{L}u = -(pu')' + qu$ where p is a differentiable function and q is continuous function such that $p(x) > 0$ in $[0,1]$. The boundary conditions $B_1(u)(0) = 0$ and $B_2(u)(1) = 0$ are called separated boundary conditions like $B_1(u)(0) = c_1u(0) = c_2u'(0)$ and $B_2(u)(1) = c_3u(1) + c_4u(1) = 0$ for some constants c_1, c_2, c_3 and c_4.

Definition 5.2.2 *A function $G(x,\xi)$ is called Green's function if it satisfies*

(1) $G(x,\xi) = G(\xi,x)$ for all $x,\xi \in [0,1]$
(2) $G(x,\xi)$ is twice differentiable for $x < \xi$ and $x > \xi$ but continuous at all $x = \xi$
(3) $G(x,\xi)$ satisfies $B_1(G) = 0$ and $B_2(G)(1) = 0$ in the variable x
(4) The jump in the derivative of G, $[\frac{\partial G}{\partial x}]\big|_{x=\xi} = \frac{-1}{p(x)}$.
(5) $\mathscr{L}(G) = 0$ for $x \neq \xi$.

We have the following existence theorem

Theorem 5.2.7 *If $\mathscr{L}(u) = 0$ in $(0,1)$, in $(0,1)$, $B_1(u)(0) = 0$, $B_2(u)(1) = 0$ has only trivial solution, then there exists a unique Green's function.*

Proof. Suppose there are two such functions $G^1(x,\xi)$ and $G^2(x,\xi)$. Then by taking $G(x,\xi) = (G^1 - G^2)(x,\xi)$, we see that G satisfies the equation for $x \neq \xi$ and $B_1(G)(0) = B_2(G)(1) = 0$. Moreover the jump in $\partial\partial G\partial x$ is 0. Now we see that

$$-pu'' = p'u' - qu$$

implies that the jump in $p\frac{\partial^2 G}{\partial^2 x}(x,\xi)$ is 0. Therefore, G satisfies $\mathscr{L}u = 0$ along with both boundary conditions. Hence $G(x,\xi) \equiv 0$. To prove the existence, let $u_1(x)$ be a solution of $\mathscr{L}u_1 = 0, B_1(u_1)(0) = 0$ and let u_2 be linearly independent from u_1 and satisfy $\mathscr{L}u_2 = 0, B_2(u)(1) = 0$. Let

$$G(x,\xi) = \begin{cases} \frac{u_1(x)u_2(\xi)}{pW} & x < \xi \\ \frac{u_1(\xi)u_2(x)}{pW} & x \geq \xi \end{cases}$$

where W is the Wronskian of u_1 and u_2. Then it i not difficult to verify that $G(x,\xi)$ satisfies $(1)-(5)$ and hence is the Green's function. □

Theorem 5.2.8 *If $\mathscr{L}(u)=0$ in $(0,1)$, $B_1(u)(0)=0$, $B_2(u)(1)=0$ has only trivial solution, Then the non-homogeneous problem*

$$\mathscr{L}(u)=f, B_1(u)(0)=0, B_2(u)(1)=0$$

has unique solution $u(x)$ given by

$$u(x)=\int_0^1 G(x,\xi)f(\xi)d\xi.$$

Proof. Proof follows by writing

$$G(x,\xi)=\begin{cases} G_1(x,\xi) & x<\xi \\ G_2(x,\xi) & x>\xi \end{cases}$$

where $G_1(.,\xi)$ satisfied the first boundary condition and $G_2(.,\xi)$ satisfies the second boundary condition along with $\mathscr{L}G_1(.,\xi)=0$, for $x<\xi$ and $\mathscr{L}G_2(.,\xi)=0$ for $x>\xi$. Writing

$$u(x)=\int_0^\xi G_1(x,\xi)f(\xi)d\xi+\int_\xi^1 G_2(x,\xi)f(\xi)d\xi.$$

Now using Newton-Leibniz formula, it is easy to see that $\mathscr{L}u=f$. □

Define the operator $K:C([0,1])\to C([0,1])$ as

$$u(x)=K(f)(x)=\int_0^1 G(x,\xi)f(\xi)d\xi$$

Then the operator K is now continuous, symmetric on $\{u\in C([0,1]),\ B_1(u)(0)=0,\ B_2(u)(1)=0\}$. Therefore, there are eigenvalues μ_i and eigen functions ϕ_i such that $\mu_i\to 0$ (if not finite). To show that $\mu_i's$ are infinite, it is enough to show that $\|K_m\|\neq 0$ for all m where

$$K_m u=\int_0^1 G_m(x,\xi)u(\xi)d\xi,\ \ G_m(x,\xi)=G(x,\xi)-\sum_{i=1}^m \mu_i\phi(x)\phi_i(\xi)$$

Suppose $\|K_m\|=0$ for some m. Let $f\in C([0,1])$ then

$$0=K_m f=Kf(x)-\sum_{i=1}^m \mu_i\phi_i(x)(f,\phi_i)$$

This implies

$$0=\mathscr{L}(0)=L(K_m f)=\mathscr{L}(Kf)-\sum_{i=1}^m \mu_i(f,\phi_i)\mathscr{L}\phi_i=f-\sum_{i=1}^m (f,\phi_i)\phi_i$$

That is

$$f(x)=\sum_{i=1}^m (f,\phi_i)\phi_i\in C^2,$$

a contradiction. □

Let μ_i be the eigenvalues and ϕ_i be the corresponding eigen function of K. Then

$$\mu_i\phi_i=K\phi_i \implies \mu_i\mathscr{L}\phi_i=\mathscr{L}K\phi_i=\phi_i$$

Therefore $\mathscr{L}\phi_i = \frac{1}{\mu_i}\phi_i$. That is, $\frac{1}{\mu_i}$ are eigenvalues and ϕ_i are eigen functions of $\mathscr{L}$. Therefore from the results of the previous section, we get the sequence of eigenvalues and eigen functions for the operator $\mathscr{L}$

Theorem 5.2.9 *Let $u \in C^2(0,1)$ such that $B_1(u)(0) = B_2(u)(1) = 0$. Then $u = \sum(u,\phi)\phi_i$ uniformly.*

Proof. We have $f = \mathscr{L}u \in C([0,1])$. That is $u = Kf$. Therefore,

$$u = Kf = \sum_{i=1}^{\infty}(Kf,\phi_i)\phi_i = \sum_{i=1}^{\infty}(u,\phi_i)\phi_i \qquad \square$$

Definition 5.2.3 *The set $\{\phi_i, i \in \mathbb{N}\}$ is called complete set if for any $f \in C([0,1])$, there exists c_i such that*

$$\|f - \sum_{i=1}^{n} c_i\phi_i\| \to 0 \text{ as } n \to \infty.$$

Finally, we have the following theorem:

Theorem 5.2.10 *Let $\{\phi_i\}$ be the sequence of eigen functions of $\mathscr{L}$, then $\{\phi_i\}$ is complete.*

Proof. Let g be a C^2 function such that $\|f-g\| < \varepsilon$. By the triangle inequality

$$\|f - \sum_{i=1}^{m}(f,\phi_i)\phi_i\| \le \|f-g\| + \|g - \sum_{i=1}^{m}(g,\phi_i)\phi_i\| + \|\sum_{i=1}^{m}(g-f,\phi_i)\phi_i\| \le 3\varepsilon$$

The last term is estimated as

$$\|\sum_{i=1}^{m}(g-f,\phi_i)\phi_i\|^2 = \sum_{i=1}^{m}|(g-f,\phi_i)|^2 \le \|f-g\|^2$$

Last inequality follows from the Bessel inequality. $\square$

Example 5.1. For a given $f \in C([0,1])$, find the Fourier series solution of

$$-u'' = f(x),\ u(0) = u(1) = 0$$

The corresponding eigen value problem is

$$-u'' = \lambda u,\ \ u(0) = u(1) = 0$$

The eigenvalues and eigen functions are

$$\lambda_n = n^2\pi^2,\ \phi_n(x) = \sin n\pi x.$$

By the above theorem we can write

$$f(x) = \sum_{i=1}^{\infty} f_i \sin n\pi x, \quad \text{where } f_i = \frac{\int_0^1 f(x)\sin n\pi x dx}{\int_0^1 \sin^2 n\pi x dx} = 2\int_0^1 f(x)\sin n\pi x dx$$

Now by taking

$$u(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x, \quad c_i = 2\int_0^1 u(x)\sin n\pi x dx.$$

If u'' has Fourier series $u''(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$. Then integration by parts (twice) and using the fact that $u(0) = u(1) = 0$, we get

$$\begin{aligned} b_n &= 2\int_0^1 u''(x)\sin n\pi x \\ &= -2n^2\pi^2 c_n \end{aligned}$$

Therefore,

$$u''(x) = -\sum_{n=1}^{\infty} c_n n^2\pi^2 \sin n\pi x.$$

Now substituting in the equation and using the fact that ϕ_i is complete set, we get

$$c_n = \frac{f_n}{n^2\pi^2}.$$

Then one can easily check that the series $\sum c_n \sin n\pi x$ converges uniformly. This method is known as Fourier series method.

Example 5.2. For a given $f \in C([0,1])$, find the Fourier series solution of

$$-u'' = f(x), \text{ in } (0,1) \;\; u'(0) = u'(1) = 0.$$

In this case the eigen values are $n^2\pi^2$ and eigen functions are $\cos n\pi x$, $n = 0, 1, 2....$ Now we can follows as above to write

$$f(x) = \sum_{n=0}^{\infty} f_n \cos n\pi x, \;\; f_i = 2\int_0^1 f(x)\cos n\pi x.$$

If $u(x) = \sum c_n \cos n\pi x$, then we can find as above

$$n^2\pi^2 c_n = f_n.$$

for $n = 0$, c_0 is arbitrary constant only when $f_0 = 0$. Otherwise there is no solution. Therefore the solution of the problem exists only when

$$\int_0^1 f(x)dx = 0.$$

In such case, solution (s) are given by

$$u(x) = C + \sum_{n=1}^{\infty} \frac{f_n}{n^2\pi^2}\cos n\pi x$$

where C is an arbitrary constant. This method can be applied for solving Initial boundary value problems

Example 5.3. For a given $f(x)$, solve the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \; x \in (0,1), \; t > 0 \\ u(x,0) &= f(x), x \in (0,1), \\ u(0,t) = 0, \;\; u(1,t) &= 0, \; t > 0. \end{aligned}$$

Taking a cue from the above examples, we write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x.$$

Then

$$c_n(t) = 2\int_0^1 u(x,t) \sin n\pi x dx.$$

Now as earlier applying integration by parts, we get

$$-u_{xx}(x,t) = \sum_{n=1}^{\infty} n^2\pi^2 c_n(t) \sin n\pi x, \quad u_t(x,t) = 2\int_0^1 c_n'(t) \sin n\pi x$$

Substituting in the equation we get

$$\sum_{n=1}^{\infty} \left(c_n'(t) + n^2\pi^2 c_n(t)\right) \sin n\pi x = 0$$

Since $\{\sin n\iota x\}$ are complete set, we get the ODE

$$c_n'(t) + n^2\pi^2 c_n(t) = 0$$

Also

$$c_n(0) = 2\int_0^1 u(x,0) \sin n\pi x = 2\int_0^1 f(x) \sin n\pi x dx = f_n$$

This initial value problem has unique solution

$$c_n(t) = f_n e^{-n^2\pi^2 t}$$

Therefore the solution is

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-n^2\pi^2 t} \sin n\pi x$$

Example 5.4. For a given $f(x)$ and $g(x)$, solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0,\ x \in (0,1),\ t > 0 \\ u(x,0) = f(x),\ u_t(x,0) &= g(x),\ \ x \in (0,1), \\ u(0,t) = 0, u(1,t) &= 0,\ t > 0. \end{aligned}$$

We follow the steps as above to write

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x.$$

In this case we see that $c_n(t)$ satisfies

$$c_n''(t) + n^2\pi^2 c_n(t) = 0,\ c_n(0) = f_n,\ c_n'(0) = g_n$$

where f_n and g_n are the Fourier coefficients of $f(x)$ and $g(x)$ respectively. These $c_n's$ are uniquely determined.

In general if $\mathscr{L}$ is a linear differential operator which has eigenvalues and eigen functions that form **complete set**, then we can find the solution the following initial boundary value problems by the method of

Fourier series. □

Problem 1: For a given $f(x), g(x) \in C([0,1])$, and $h(x,t) \in C([0,1]\times[0,\infty))$ find a solution of the IBVP of the parabolic equation:

$$\begin{aligned} u_t - \mathscr{L}(u) &= h(x,t),\ x \in (0,1),\ t > 0 \\ u(x,0) &= f(x), x \in (0,1), \\ u(0,t) &= 0,\ u(1,t) = 0,\ t > 0. \end{aligned}$$

Problem 2: For a given $f(x), g(x) \in C([0,1])$ and $h(x,t) \in C([0,1]\times[0,\infty))$, find a solution of IBVP of the wave equation:

$$\begin{aligned} u_{tt} - \mathscr{L}(u) &= h(x,t),\ x \in (0,1),\ t > 0 \\ u(x,0) &= f(x),\ u_t(x,0) = g(x),\ \ x \in (0,1), \\ u(0,t) &= 0,\ u(1,t) = 0,\ t > 0. \end{aligned}$$

We can also solve the elliptic problems on rectanglular domains using this method. For example

Example 5.5. Consider the problem on the square $\Omega = [0,1]\times[0,1]$.

$$-u_{xx} - u_{yy} = f(x,y) \text{ in } \Omega, \quad u = 0 \text{ on the lines } x = 0,\ x = 1 \text{ and } y = 0,\ y = 1.$$

As earlier, we first consider the eigenvalue problem. That is,

$$-u_{xx} - u_{yy} = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on the lines } x = 0,\ x = 1 \text{ and } y = 0,\ y = 1.$$

By separating the variables, taking $u(x,y) = \sum_{j=1}^{\infty} w_j(y) v_j(x) = \sum_{j=1}^{\infty} w_j(y)\sin(j\pi x)$ we get

$$\sum_{j=1}^{\infty} (j^2\pi^2 w_j + (w_j)_{yy})\sin(j\pi x) = \sum_{j=1}^{\infty} \lambda w_j(y)\sin(j\pi x)$$

This implies ($\{\sin j\pi x\}_j$ forms basis of L^2)

$$(w_j)_{yy} = (\lambda - \pi^2 j^2) w_j,\ w_j(0) = 0,\ w_j(1) = 0.$$

Solving this we get

$$w_k(y) = \sin(k\pi y),\ \text{provided } \lambda - j^2\pi^2 = k^2\pi^2,\ k = 1,2,...$$

Therefore the eigenvalues are $\lambda_{jk} = (j^2+k^2)\pi^2, j,k = 1,2,3...$ and eigen functions are $w_{jk} = \sin(j\pi y)\sin(k\pi x)$. Then we can write the eigen function expansion of $f(x,y)$ as

$$f(x,y) = \sum_{j,k=1}^{\infty} f_{jk}\pi^2 \sin(j\pi x)\sin(k\pi y), \ \text{ where } f_{jk} = 4\int_0^1\int_0^1 f(x,y)\sin j\pi x \sin k\pi y dxdy$$

are the Fourier coefficients. Then by writing $u(x,y)$ as

$$u(x,y) = \sum_{j,k=1}^{\infty} c_{j,k}\sin(j\pi x)\sin(k\pi y)$$

and substituting in the equation, we get

$$c_{jk}(j^2+k^2)\pi^2 = f_{jk}$$

The problem has unique solution

$$u(x,y) = \sum_{j,k=1}^{\infty} \frac{f_{jk}}{(j^2+k^2)\pi^2} \sin(j\pi x)\sin(k\pi y).$$

Singular SLP problems do not require boundary condition to get the symmetry of the corresponding integral operator. For example

$$\mathscr{L}(u) = -\left((1-x^2)u'\right)' \ \ x \in (-1,1)$$

It is easy to verify (using integration by parts) that

$$(\mathscr{L}u,v) = (u,\mathscr{L}v), \ \ u,v \in C([-1,1])$$

Therefore the operator $\mathscr{L}$ is self-adjoint on $C([-1,1])$. The corresponding eigenvalue problem

$$\mathscr{L}u = \lambda u$$

has eigenvalues $n(n+1)$ and eigenfunctions are the Legendre polynomials $P_n(x)$. Also from the well approximation theorems we know that every continuous function can be uniformly approximated by Legendre polynomials.So the following

Example 5.6. Solve the problem

$$-\left((1-x^2)u'\right)' = f(x), \ x \in [-1,1]$$

Writing $f(x) = \sum f_n P_n(x)$, where, $f_n = \frac{\int_{-1}^{1} f(x)P_n(x)dx}{\int_{-1}^{1} P_n^2(x)dx}$ and $u(x) = \sum c_n P_n(x)$ and substituting in the equation, we get

$$c_n n(n+1) = f_n$$

from this one gets c_n uniquely. □
Similarly, Bessel functions, Hermite and Laguerre polynomials are sued to find solutions which are also known as Fourier Series solutions.

From above examples we see that we can find the eigenvalues and eigen functions in dimension one easily. In case of two or higher dimensions we can find these eigen values and eigen functions only in special types of domains like rectangles. In general, if Ω is any bounded domain in $\mathbb{R}^n$, then we may not be able to find eigenfunctions explicitly. So it would be interesting to study if such eigenvalues and eigen functions exist. We have seen that this requires the understanding of Green's function for such domains. In the next section we will study the Green's function for the Laplacian operator.

5.2.2 Unbounded domains

consider the heat condution in a semi-infinite insulated rod with uniform cross section with forced boundary condition at one end and no internal sources.

$$u_t = ku_{xx},\ 0 < x < \infty,\ t > 0$$
$$u(0,t) = 0,\ t > 0,\ u(x,t) \text{ bounded as } x \to \infty$$
$$u(x,0) = f(x), x > 0.$$

From the above discussion, we get the following eigenvalue problem

$$\phi'' = -\lambda^2 \phi,\ x > 0$$
$$\phi(0) = 0,\ \phi(x) \text{ bounded as } x \to \infty.$$

Then we see that for every $\lambda > 0$, $\phi_\lambda(x) = C_\lambda \sin(\lambda x), C_\lambda \in \mathbb{R}$ is a nontrivial solution. Therefore by taking

$$u(x,t) = \phi_\lambda(x)T(x)$$

we see that

$$T(t) = e^{-\lambda^2 kt}.$$

Therefore we have a continuum of functions satisfying the heat equation and boundary conditions

$$u(x,t,\lambda) = C_\lambda \sin(\lambda x) e^{-\lambda^2 kt}$$

since the equation is linear, by the superposition principle we must use an integral–the continuous analogue of a sum or series. Thus a candidate for the solution $u(x,t)$ have the form

$$u(x,t) = \int_0^\infty B(\lambda)\sin(\lambda x) e^{-\lambda^2 kt} d\lambda$$

Then the initial condition implies

$$f(x) = u(x,0) = \int_0^\infty B(\lambda)\sin(\lambda x) d\lambda,\ x > 0.$$

If such $B(\lambda)$ exists then by the fourier integral theorem, we recongnize $B(\lambda)$ as

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx,$$

Theorem 5.2.11 *Fourier Integral Theorem: Let $f(x)$ be a piecewise continuous and integrable function on $\mathbb{R}$. Then*

$$\int_0^\infty A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) d\lambda = \frac{1}{2}[f(x^+) + f(x^-)],\ x \in \mathbb{R}$$

where

$$A(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} f(x)\cos(\lambda x)dx,\ \ B(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} f(x)\sin(\lambda x)dx.$$

From the above theorem it is easy to see that if $f(x)$ is odd function then $A(\lambda) = 0$ and $B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx$. If $f(x)$ is even function then $B(\lambda) = 0$ and $A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)dx$. So given any function defined on $(0,\infty)$, we can extend it as odd function OR even function. This leads to the definition of Fourier sine integral and Fourier cosine integral.

Example 5.7. Solve the IBVP of heat equation on semi-infinite rod

$$\begin{aligned} u_t =& ku_{xx},\ 0 < x < \infty,\ t > 0 \\ u(0,t) =& 0,\ t > 0,\ u(x,t) \text{ bounded as } x \to \infty \\ u(x,0) =& \begin{cases} A_0 & 0 < x < 1, \\ 0 & x \geq 1. \end{cases} \end{aligned}$$

This means a unit length of the rod has initial temparature A_0 and then it is 0 for the rest. Then solution is given by

$$u(x,t) = \int_0^\infty B(\lambda)\sin(\lambda x)e^{-\lambda^2 kt}d\lambda$$

where

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx = \frac{2A_0}{\lambda\pi}(1-\cos(\lambda b)).$$

Hence

$$u(x,t) = \frac{2A_0}{\pi}\int_0^\infty \frac{(1-\cos(\lambda b))}{\lambda}\sin(\lambda x)e^{-\lambda^2 kt}d\lambda.$$

□

In case of the the domain is whole $\mathbb{R}$, then the problem looks like

Example 5.8. Consider the initial value problem of the heat equation in the whole $\mathbb{R}$:

$$\begin{aligned} u_t =& ku_{xx},\ x \in \mathbb{R},\ t > 0 \\ |u(x,t)|& \text{ bounded as } x \to \pm\infty \\ u(x,0) =& f(x), x > 0. \end{aligned}$$

In this case we get the eigenvalue problem as

$$\begin{aligned} \phi'' = -\lambda^2\phi,\ x \in \mathbb{R} \\ \phi(x) \text{ bounded as } x \to \pm\infty. \end{aligned}$$

It is easy to see that every $\lambda > 0$ is an eigenvalue and $A_\lambda\cos(\lambda x), B_\lambda\sin(\lambda x)$ where $A_\lambda, B_\lambda \in \mathbb{R}$, are eigen functions. Here, we have

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)d\lambda,\ x \in \mathbb{R}.$$

At time $t = 0$, we get

$$f(x) = \int_0^\infty A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)d\lambda$$

Therefore,

$$A(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} f(x)\cos(\lambda x)dx,\ \ B(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} f(x)\sin(\lambda x)dx.$$

Next we consider the wave equation in $(0,\infty)$:

Example 5.9. consider the initial boundary value problem for the wave equation on $(0,\infty)$:

$$
\begin{aligned}
u_{tt} =& k u_{xx},\ 0 < x < \infty,\ t > 0 \\
u(0,t) = 0,\ t > 0,\ &|u(x,t)| \text{ bounded as } x \to \infty \\
u(x,0) = f(x),\ u_t&(x.0) = g(x),\ x \in (0,\infty).
\end{aligned}
$$

In this case by writing $u(x,t) = T(t)\phi(x)$ we get

$$
\begin{aligned}
\phi^{''} + \lambda^2\phi = 0, x > 0 \\
T^{''} + \lambda^2 kT = 0,\ t > 0 \\
\phi(0) = 0, |\phi(x)| \text{ is bounded.}
\end{aligned}
$$

The solutions are

$$\phi_\lambda(x) = \sin(\lambda x),\ T_\lambda(t) = A_\lambda \cos(\lambda\sqrt{k}t) + B_\lambda \sin(\lambda\sqrt{k}t),$$

Therefore

$$u(x,t) = \int_0^\infty [A_\lambda \cos(\lambda\sqrt{k}t) + B_\lambda \sin(\lambda\sqrt{k}t)] \sin(\lambda x) d\lambda.$$

Then A_λ, B_λ are determined from the initial conditions

$$u(x,0) = f(x) = \int_0^\infty A_\lambda \sin(\lambda x) d\lambda, x > 0,$$

$$u_t(x,0) = g(x) = \int_0^\infty \lambda\sqrt{k} B_\lambda \sin(\lambda x) d\lambda, x > 0,$$

By the Fourier integral theorem, we get

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx,\ B(\lambda) = \frac{2}{\pi\lambda\sqrt{k}}\int_0^\infty g(x)\sin(\lambda x)dx.$$

so if $f(x)$ is integrable, the both these integrals exist.

5.2.3 Exercises

1. Solve the following BVPs using eigenfunction expansions, shifting the data if necessary.
   $(a)\ -u'' = 1\ \ u(0) = u(1) = 0 \quad (b)\ -u'' = e^x\ u(0) = u(1) = 1$
   $(c)\ -u'' = x,\ u(0) = u(1) = 0 \quad (d)\ -u'' + 2u = 1,\ u(0) = u(1) = 0$
   $(e)\ -u'' = x^2,\ u(0) = u(1) = 0 \quad (f)\ -u'' + 2u = \frac{1}{2} - x,\ u(0) = u(2) = 0.$
2. Solve the following BVPs using eigenfunction expansion method:
   $(i)\ -u'' = f(x), 0 < x < l,\ u'(0) = a,\ u'(l) = b \quad (ii)\ u'' + u = f(x),\ u'(0) = a, u'(l) = b.$
3. Solve the following IBVP using the Fourier series method:
   $(a)\ u_t - u_{xx} = 0, 0 < x < 1, t > 0$, $u(x,0) = x,\ 0 < x < 1,$ $u(0,t) = 0,\ u(1,t) = 0, t > 0.$
   $(b)\ u_t - u_{xx} = \sin t, 0 < x < 1, t > 0$, $u(x,0) = x,\ 0 < x < 1,$ $u(0,t) = 0,\ u(1,t) = 0, t > 0.$
4. Solve the IBVP by shifting the data:

$$\begin{aligned} u_t - u_{xx} &= 0, 0 < x < 1, t > 0 \\ u(x,0) &= x,\ 0 < x < 1, \\ u(0,t) &= 0,\ u(1,t) = \cos t, t > 0. \end{aligned}$$

5. Solve the IBVP using Fourier series method
   (a) $u_t - u_{xx} = 0, 0 < x < 1, t > 0$, $u(x,0) = x(1-x),\ 0 < x < 1$, $\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(1,t) = 0, t > 0.$
   (b) $u_t - u_{xx} = \frac{1}{2} - x, 0 < x < 1, t > 0$, $u(x,0) = x(1-x),\ 0 < x < 1$, $\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(1,t) = 0, t > 0.$
6. Solve the BVP using the Fourier series method:

$$-u'' = \frac{1}{10}x(x^2 - \pi^2),\ -\pi < x < \pi,\ u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi)$$

7. Solve the IBVP using the Fourier series method

$$\begin{aligned} u_t - u_{xx} &= \frac{1}{10}x(x^2 - \pi^2), -\pi < x < \pi, t > 0 \\ u(x,0) &= 25,\ -\pi < x < \pi, \\ \frac{\partial u}{\partial t}(-\pi,t) &= \frac{\partial u}{\partial t}(\pi,t),\ u(-\pi,t) = u(\pi,t), t > 0. \end{aligned}$$

8. Solve the IBVP using Fourier series method
   (a) $u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0$, $u(x,0) = x(1-x),\ 0 < x < l$, $\frac{\partial u}{\partial t}(x,0) = x,\ u(0,t) = u(l,t) = 0, t > 0.$
   (b) $u_{tt} - c^2 u_{xx} = 100, 0 < x < l, t > 0$, $u(x,0) = x(1-x),\ 0 < x < l$, $\frac{\partial u}{\partial t}(x,0) = x,\ u(0,t) = u(l,t) = 0, t > 0.$
9. Let Ω be a unit square in $\mathbb{R}^2$, and let $f : \Omega \to \mathbb{R}$ be $f(x) = x_1 x_2 (\frac{1}{4} - x_1)(1 - x_1)(1 - x_2)$. Then solve the BVP

$$-\Delta u = f(x) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

10. Repeat the above exercise with $f(x)$ replaced by $f(x) = x_1(1 - x_1)^2$.
11. Solve the BVP: $-\Delta u = f(x)$ in $B_1(0)$, $u = 0$ on $\partial B_1(0)$ with
    (i) $f(x) = 1 - \sqrt{x_1^2 - x_2^2}$ (ii) $f(x) = \sqrt{x_1^2 + x_2^2}$.
12. Solve the following BVP
    $\Delta u = 0, \quad 0 \le x \le a, 0 \le y \le b$ satisfying the boundary conditions
    $u(0,y) = 0,\ u(x,0) = 0, u(x,b) = 0, \dfrac{\partial u}{\partial x} = \sin^3 \dfrac{\pi y}{a}.$
13. The boundaries of the rectangle $0 \le x \le a, 0 \le y \le b$ are maintained at zero temperature. If at $t = 0$ the temperature T has the prescribed value $f(x,y)$, then find the temperature at a point $t > 0$, within the rectangle.
14. Find the solution of the wave equation $u_{tt} = c^2 u_{xx}$ under the following conditions:
    $u(0,t) = u(L,t) = 0,\ u_t(x,0) = 0, u(x,0) = \begin{cases} \frac{\varepsilon x}{b} & 0 \le x \le b \\ \frac{\varepsilon(x-L)}{(b-L)} & b \le x \le L. \end{cases}$
15. Find a Fourier integral formula for the solution of the problem:

$$\begin{aligned} u_t &= k u_{xx},\ x > 0, t > 0 \\ u_x(0,t) &= 0,\ t > 0 \\ u(x,0) &= f(x),\ x > 0. \end{aligned}$$

16. Find a Fourier integral formula for the solution of the problem:

$$\begin{aligned} u_t &= k u_{xx},\ x > 0, t > 0 \\ u_x(0,t) &= T_0,\ t > 0 \\ u(x,0) &= T_0(1 - e^{-\alpha x}),\ x > 0. \end{aligned}$$

17. Find a Fourier integral formula for the solution of the problem:

$$\begin{aligned} u_t &= k u_{xx},\ x \in \mathbb{R}, t > 0 \\ u(x,0) &= e^{-\alpha|x|},\ x \in \mathbb{R}. \end{aligned}$$

18. Find a Fourier integral formula for the solution of the problem:

$$\begin{aligned} u_{tt} &= k u_{xx},\ x \in \mathbb{R}, t > 0 \\ u(x,0) &= f(x),\ u_t(x,0) = 0,\ x \in \mathbb{R}. \end{aligned}$$