A brief review of

Several Variable Differential Calculus

1 A brief review

We recall the following elements of Several variable calculus which are essential for understanding the first order Partial differential equations.

Definition 1.0.1. *Let Ω be a open set in $\mathbb{R}^2$, $(a,b) \in \Omega$ and let f be a real valued function defined on Ω except possibly at (a,b). Then the limit $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \epsilon.$$

Example 1.0.2. *Finding limit through polar coordinates:*
Consider the function $f(x,y) = \dfrac{x^3}{x^2+y^2}$.

This function is defined in $\mathbb{R}^2 \backslash \{(0,0)\}$. Taking $x = r\cos\theta, y = r\sin\theta$, we get

$$|f(r,\theta)| = |r\cos^3\theta| \le r \to 0 \ \ as \ \ r \to 0.$$

Example 1.0.3. *Example of function which has different limits along different straight lines.*
Consider the function $f(x,y) = \dfrac{xy}{x^2+y^2}$.

Then along the straight lines $y = mx$, we get $f(x,mx) = \frac{m}{1+m^2}$. Hence limit does not exist.

Example 1.0.4. *Example of function which has different limits along different curves:*
Consider the function $f(x,y)$:

$$f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2} & (x,y) \not\equiv (0,0) \\ 0 & (x,y) \equiv (0,0) \end{cases}$$

Then along the curves $y = mx^2$, we get $f(x,mx^2) = \frac{1-m^2}{1+m^2}$. Hence limit does not exist.

Example 1.0.5. *Example function where polar coordinates seem to give wrong conclusions*
Consider the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$.

Taking the path, $y = mx^2$, we see that the limit does not exist at $(0,0)$. Now taking $x = r\cos\theta, y = r\sin\theta$, we get

$$f(r,\theta) = \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

For any $r > 0$, the denominator is > 0. For each fixed r and taking $\theta \to 0$, we see it tends to 0. Since $|\cos^2\theta \sin\theta| \leq 1$, we tend to think for a while that this limit goes to zero as $r \to 0$. But along the path $r\sin\theta = r^2\cos^2\theta$, (i.e., $r = \frac{\sin\theta}{\cos^2\theta}$), we get

$$f(r,\theta) = \frac{2\sin^2\theta}{2\sin^2\theta} = 1.$$

Therefore the limit does not exist.

Let f be a real valued function defined in a ball around (a,b). Then

Definition 1.0.6. *f is said to be continuous at (a,b) if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Example 1.0.7. *The function*

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}$$

Let $\epsilon > 0$. Then $|f(x,y) - 0| = |x|\frac{|y|}{\sqrt{x^2+y^2}} \leq |x|$. So if we choose $\delta = \epsilon$, then $|f(x,y)| \leq \epsilon$. Therefore, f is continuous at $(0,0)$.

Partial Derivatives: The partial derivative of f with respect to x at (a,b) is defined as

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{1}{h}\left(f(a+h,b) - f(a,b)\right).$$

similarly, the partial derivative with respect to y at (a,b) is defined as

$$\frac{\partial f}{\partial y}(a,b) = \lim_{k\to 0}\frac{1}{k}\left(f(a,b+k) - f(a,b)\right).$$

Example 1.0.8. *Consider the function*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \not\equiv (0,0) \\ 0 & (x,y) \equiv (0,0) \end{cases}$$

As noted earlier, this is not a continuous function, but

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

Similarly, we can show that $f_y(0,0)$ exists.

Also for a continuous function, partial derivatives need not exist. For example $f(x,y) =$

$|x|+|y|$. This is a continuous function at $(0,0)$. Indeed, for any $\epsilon>0$, we can take $\delta<\epsilon/2$. But partial derivatives do not exist at $(0,0)$.

Theorem 1.0.9. ***Sufficient condition for continuity:*** *Suppose one of the partial derivatives exist at (a,b) and the other partial derivative is bounded in a neighborhood of (a,b). Then $f(x,y)$ is continuous at (a,b).*

Proof. Let f_y exists at (a,b). Then

$$f(a,b+k)-f(a,b)=kf_y(a,b)+\epsilon_1 k,$$

where $\epsilon_1\to 0$ as $k\to 0$. Since f_x exists and bounded in a neighborhood of at (a,b),

$$\begin{aligned}
f(a+h,b+k)-f(a,b) =&f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b)\\
=&hf_x(a+\theta h,b+k)+kf_y(a,b)+\epsilon_1 k\\
\leq&hM+k|f_y(a,b)|+\epsilon_1 k\\
&\to 0 \text{ as } h,k\to 0.
\end{aligned}$$

Directional derivatives, Definition and examples

Let $\hat{p}=p_1\hat{i}+p_2\hat{j}$ be any **unit vector**. Then the directional derivative of $f(x,y)$ at (a,b) in the direction of $\hat{p}$ is

$$D_{\hat{p}}f(a,b)=\lim_{s\to 0}\frac{f(a+sp_1,b+sp_2)-f(a,b)}{s}.$$

Example 1.0.10. *$f(x,y)=x^2+xy$ at $P(1,2)$ in the direction of unit vector $p=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$.*

$$\begin{aligned}
D_{\hat{p}}f(1,2)&=\lim_{s\to 0}\frac{f(1+\frac{s}{\sqrt{2}},2+\frac{s}{\sqrt{2}})-f(1,2)}{s}\\
&=\lim_{s\to 0}\frac{1}{s}\left(s^2+s(2\sqrt{2}+\frac{1}{\sqrt{2}})\right)=2\sqrt{2}+\frac{1}{\sqrt{2}}
\end{aligned}$$

The existence of partial derivatives does not guarantee the existence of directional derivatives in all directions. For example take

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x=y=0\end{cases}.$$

Let $\overrightarrow{p}=(p_1,p_2)$ such that $p_1^2+p_2^2=1$. Then the directional derivative along p is

$$D_{\hat{p}}f(0,0)=\lim_{h\to 0}\frac{f(hp_1,hp_2)-f(0,0)}{h}=\lim_{h\to 0}\frac{p_1p_2}{h(p_1^2+p_2^2)}$$

exist if and only if $p_1 = 0$ or $p_2 = 0$.

The existence of all directional derivatives does not guarantee the continuity of the function. For example

Example 1.0.11.
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \not\equiv (0,0) \\ 0 & x = y = 0 \end{cases}.$$

Let $\overrightarrow{p} = (p_1, p_2)$ such that $p_1^2 + p_2^2 = 1$. Then the directional derivative along p is

$$\begin{aligned} D_{\widehat{p}} f(0,0) &= \lim_{s\to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s} \\ &= \lim_{s\to 0} \frac{s^3 p_1^2 p_2}{s(s^4 p_1^4 + s^2 p_2^2)} \\ &= \frac{p_1^2 p_2}{p_2^2} \text{ if } p_2 \neq 0 \end{aligned}$$

In case of $p_2 = 0$, we can compute the partial derivative w.r.t y to be 0. Therefore all the directional derivatives exist. But this function is not continuous ($y = mx^2$ and $x \to 0$).

Differentiability: Let D be an open subset of $\mathbb{R}^2$. Then

Definition 1.0.12. *A function $f(x,y) : D \to \mathbb{R}$ is differentiable at a point (a,b) of D if there exists $\epsilon_1 = \epsilon(h,k), \epsilon_2 = \epsilon_2(h,k)$ such that*

$$f(a + hb + k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\epsilon_1 + k\epsilon_2,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$.

Example 1.0.13. *Consider the function $f(x,y) = x^2+y^2+xy$. Then $f_x(0,0) = f_y(0,0) = 0$. Also*
$$f(h,k) - f(0,0) = h^2 + k^2 + hk = 0h + 0k + \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1 = h + k, \epsilon_2 = k$. Therefore f is differentiable at $(0,0)$.

Example 1.0.14. *Show that the following function $f(x,y)$ is is not differentiable at $(0,0)$*

$$f(x,y) = \begin{cases} x \sin\frac{1}{x} + y \sin\frac{1}{y}, & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Using the boundedness of sin and cos, we get $|f(x,y)| \leq |x| + |y| \leq 2\sqrt{x^2+y^2}$ implies that

f is continuous at $(0,0)$. Also

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0.$$

$$f_y(0,k) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$$

If f is differentiable, then there exists ϵ_1, ϵ_2 such that

$$f(h,k) - f(0,0) = \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1, \epsilon_2 \to 0$ as $h, k \to 0$. Now taking $h = k$, we get

$$f(h,h) = (\epsilon_1 + \epsilon_2)h \implies 2h \sin\frac{1}{h} = h(\epsilon_1 + \epsilon_2).$$

So as $h \to 0$, we get $\sin\frac{1}{h} \to 0$, a contradiction.
Equivalent condition for differentiability:

Theorem 1.0.15. *f is differentiable at (a,b) $\iff \lim\limits_{\rho\to 0} \dfrac{\Delta f - df}{\rho} = 0$, where $\Delta f = f(a+h, b+k) - f(a,b)$, $df = hf_x(a,b) + kf_y(a,b)$ and $\rho = \sqrt{h^2+k^2}$*

Example 1.0.16. *Consider $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \not\equiv 0 \\ 0 & x = y = 0 \end{cases}$.*

Partial derivatives exist at $(0,0)$ and $f_x(0,0) = f_y(,0) = 0$. By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we get

$$\frac{\Delta f - df}{\rho} = \frac{h^2k}{\rho^3} = \frac{\rho^3\cos^2\theta\sin\theta}{\rho^3} = \cos^2\theta\sin\theta.$$

The limit does not exist. Therefore f is NOT differentiable at $(0,0)$.

Theorem 1.0.17. ***Sufficient Condition:*** *Suppose $f_x(x,y)$ and $f_y(x,y)$ exist in an open neighborhood containing (a,b) and both functions are continuous at (a,b). Then f is differentiable at (a,b).*

There are functions which are Differentiable but the partial derivatives need not be continuous. For example,

Example 1.0.18. *consider the function*

$$f(x,y) = \begin{cases} x^3\sin\frac{1}{x^2} + y^3\sin\frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Then

$$f_x(x,y) = \begin{cases} 3x^2 \sin \frac{1}{x^2} - 2\cos \frac{1}{x^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Also $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$. So partial derivatives are not continuous at $(0,0)$.

$$\begin{aligned} f(\Delta x, \Delta y) &= (\Delta x)^3 \sin \frac{1}{(\Delta x)^2} + (\Delta y)^3 \sin \frac{1}{(\Delta y)^2} \\ &= 0 + 0 + \epsilon_1 \Delta x + \epsilon_2 \Delta y \end{aligned}$$

where $\epsilon_1 = (\Delta x)^2 \sin \frac{1}{(\Delta x)^2}$ and $\epsilon_2 = (\Delta y)^2 \sin \frac{1}{(\Delta y)^2}$. It is easy to check that $\epsilon_1, \epsilon_2 \to 0$. So f is differentiable at $(0,0)$.

There are continuous functions for which directional derivatives exist in any direction, but the function is not differentiable.

Example 1.0.19. *Consider the function*

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

(Exercise problem)

Chain rule:

Partial derivatives of composite functions: Let $z = F(u,v)$ and $u = \phi(x,y), v = \psi(x,y)$. Then $z = F(\phi(x,y), \psi(x,y))$ as a function of x, y. Suppose F, ϕ, ψ have continuous partial derivatives, then we can find the partial derivatives of z w.r.t x, y as follows: Let x be increased by Δx, keeping y constant. Then the increment in u is $\Delta_x u = u(x+\Delta x, y) - u(x,y)$ and similarly for v. Then the increment in z is (as z is differentiable as a function of u, v)

$$\Delta_x z := z(x+\Delta x, y+\Delta y) - z(x,y) = \frac{\partial F}{\partial u}\Delta_x u + \frac{\partial F}{\partial v}\Delta_x v + \epsilon_1 \Delta_x u + \epsilon_2 \Delta_x v$$

Now dividing by Δx

$$\frac{\Delta_x z}{\Delta x} = \frac{\partial F}{\partial u}\frac{\Delta_x u}{\Delta x} + \frac{\partial F}{\partial v}\frac{\Delta_x v}{\Delta x} + \epsilon_1 \frac{\Delta_x u}{\Delta x} + \epsilon_2 \frac{\Delta_x v}{\Delta x}$$

Taking $\Delta x \to 0$, we get

$$\begin{aligned}\frac{\partial z}{\partial x} =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + (\lim_{\Delta x \to 0} \epsilon_1)\frac{\partial u}{\partial x} + (\lim_{\Delta x \to 0} \epsilon_2)\frac{\partial v}{\partial x} \\ =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\end{aligned}$$

similarly, one can show

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

Example 1.0.20. *Let* $z = \ln(u^2 + v^2), u = e^{x+y^2}, v = x^2 + y$.
Then $z_u = \frac{2u}{u^2+v}, z_v = \frac{1}{u^2+v}, u_x = e^{x+y^2}, v_x = 2x$.

$$z_x = \frac{2u}{u^2+v}e^{x+y^2} + \frac{2x}{u^2+v}$$

Theorem 1.0.21. *If* $f(x, y)$ *is differentiable, then the directional derivative in the direction* $\hat{p}$ *at* (a, b)*is*

$$D_{\hat{p}}f(a, b) = \nabla f(a, b) \cdot \hat{p}.$$

Proof. Let $\hat{p} = (p_1, p_2)$. Then from the definition,

$$\lim_{s\to 0} \frac{f(a + sp_1, b + sp_2) - f(a, b)}{s} = \lim_{s\to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s}$$

where $x(s) = a + sp_1, y(s) = b + sp_2$.
From the chain rule,

$$\lim_{s\to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s} = \frac{\partial f}{\partial x}(a, b)\frac{dx}{ds} + \frac{\partial f}{\partial y}(a, b)\frac{dy}{ds} = \nabla f(a, b) \cdot (p_1, p_2).$$

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The above proposition is again only sufficient condition. That is The formula $D_{\hat{p}}f = \nabla f \cdot \hat{p}$ can still hold even when function f is NOT differentiable. for example

$$f(x, y) = \begin{cases} \dfrac{x^2 y\sqrt{|y|}}{x^4 + y^2} & (x, y) \not\equiv (0, 0) \\ 0 & (x, y) \equiv (0, 0) \end{cases}$$

In this case it is easy to check from the definition that all directional derivatives at the origin are equal to zero. But the function is not differentiable at the origin. To show this take the polar coordinates $x = r\cos\theta, y = r\sin\theta$ to see that

$$\frac{\Delta f - df}{r} = \sqrt{r}\,\frac{\sin^2\theta\cos\theta\sqrt{|\sin\theta|}}{r^2\cos^2\theta + \sin^2\theta}.$$

Taking $r = \frac{\sin\theta}{\cos\theta}$ and taking $\theta \to 0$, we see that the above limit approaches infinity. ///

Suppose a smooth curve γ is $\mathbb{R}^2$ is defined as $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ for $t \in [a, b]$. Then the rate of change on the surface $z = f(x, y)$ along $r(t)$ can be seen by chain rule as

$$\frac{d}{dt}f(r(t)) = \frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot r'(t).$$

2 Calculus of Vector valued functions

Definition 2.0.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ be well defined functin in a neighbourhood of a point $A \in \mathbb{R}^n$. Then f is said to be differentiable at A if there exists a linear transformation $T : \mathbb{R}^n$ to $\mathbb{R}^m$ such that*

$$\lim_{\|h\| \to 0} \frac{\|f(A + h) - f(A) - Th\|}{\|h\|} = 0.$$

If a function is f is differentiable at a point A, then it is not difficult to see that

$$T = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

We may also denote $f'(A) = T$. All the elements defined for realvalued functions can be generalized to this case.

Mean Value theoem: Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$. Then we can apply the meanvalue theorem of one variable to each f_i to see

$$f_i(x + h) - f_i(x) = \nabla f_i(x + t_i h) \cdot h$$

But there will not be one $t*$ for all t_i, $i = 1, 2, 3, ...m$. For exaple

$$f : [0, 2\pi]\mathbb{R}^2; f(x) = (\cos x, \sin x)$$

then $f(2\pi) - f(0) = 0$ but $f_1'(x) = -\sin x, f_2'(x) = \cos x$ are never simultaneously zero as x varies over $[0, 2\pi]$. So the Mean Value theorem takes the following form:

Theorem 2.0.2. ***Mean Value Inequality:*** *Suppose f is differentiable in an open convex set D around the point x and $\|f'(y)\| \leq M$ for all $y \in D$, for some $M > 0$. Then*

$$\|f(x + h) - f(x)\| \leq M\|h\|$$

Proof. Define $g_i(t) = f_i(x + th), t \in [0, 1]$. Then we have

$$f_i(x+h) - f_i(x) = g_i(1) - g_i(0) = \int_0^1 g_i'(s)ds = \int_0^1 \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x+th)h_j \right) dt \qquad (2.1)$$

Now using Cauchy-Scwartz inequality one obtains the following for any vector valued function $v(t)$

$$\| \int_0^1 v(t)dt \| \leq \int_0^1 \|v(t)\| dt$$

Indeed, taking $u = \int_0^1 v(t)dt$

$$\|u\|^2 = \langle u, u \rangle = \left\langle \int_0^1 v(t)dt, u \right\rangle = \int_0^1 \langle v(t), u \rangle dt \leq \int_0^1 \|v(t)\| \|u\| \leq \|u\| \int_0^t \|v(t)\| dt$$

From (2.1), we get

$$f(x+h) - f(x) = \left(\int_0^1 f'(x+th)dt \right) \cdot h$$

where f' is the matrix T described above. Therefore,

$$\|f(x+h) - f(x)\| \leq M\|h\|.$$

Remark 2.1. *It is worthy to note that the the equality many not hold here as in the one variable case.*

Inverse function theorem:

Theorem 2.0.3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 in an open set around $A \in \mathbb{R}^n$ and $Det(f'(A)) \neq 0$, then there exists an open ball $B_r(A)$ such that*

1. *f has C^1 inverse function f^{-1} in $B_r(A)$*
2. *$(f^{-1})'(f(x)) = (f'(x))^{-1}$ for all $x \in B_r(A)$.*

In simple terms, if $u = f_1(x, y), v = f_2(x, y)$ such that f_1, f_2 has continuous partial derivates in a neigbourhood around (x_0, y_0) and let $Det(f'(x_0, y_0)) \neq 0$, then there exists $r > 0$ such that in $B_r((x_0, y_0))$ there is $G(u, v) = (g_1(u, v), g_2(u, v))$ such that $x = g_1(u, v), y = g_2(u, v)$. The function G is called the inverse of F. Moreover

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}^{-1}$$

Example 2.0.4. *Consider the function $f(x, y) = (u, v) = (x \cos y, x \sin y)$ then near $(x_0, y_0), x_0 \neq 0$, (x, y) can be expressed as differentiable function of (u, v).*

In this case, $u_x = \cos y, u_y = -x\sin y, v_x = \sin y, v_y = x\cos y$. *Clearly all of these are continuous everywhere.*

$$Det(f'(x_0,y_0)) = Det\begin{pmatrix} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \end{pmatrix}\Bigg|_{(x_0,y_0)} = Det\begin{pmatrix} \cos y_0, -x_0\sin y_0 \\ \sin y_0, x_0\cos y_0 \end{pmatrix} = x_0 \neq 0.$$

Hence by the inverse function theorem f *has* C^1 *inverse and* (x,y) *can be expressed as functions of* (u,v). *Moreover,*

$$(f^{-1})'(f(x)) = (f'(x))^{-1} \implies$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \end{pmatrix}^{-1} = \begin{pmatrix} \cos y_0 & -x_0\sin y_0 \\ \sin y_0 & x_0\cos y_0 \end{pmatrix}^{-1} = \frac{1}{x_0}\begin{pmatrix} x_0\cos y_0 & x_0\sin y_0 \\ -\sin y_0 & \cos y_0 \end{pmatrix}$$

By comparing, we have

$$\frac{\partial x}{\partial u} = \cos y_0, \quad \frac{\partial x}{\partial v} = \sin y_0$$

Just to remark that in the above $\frac{\partial x}{\partial u} = \cos y_0 \neq \frac{1}{\cos y_0} = \frac{1}{\frac{\partial u}{\partial x}}$. *So there is no inverse partial differentiation.*

Example 2.0.5. *Let* $u = x^2 - y$ *and* $v = x - y$. *Then to check if the inverse exists*

$$|J| = Det\begin{pmatrix} 2x & -1 \\ 1 & -1 \end{pmatrix} = 1 - 2x$$

So this function is invertible at all points other than $x = \frac{1}{2}$. *That is, if* $x \neq \frac{1}{2}$ *we can express* (x,y) *as a* C^1 *function of* (u,v).

Remark 2.2. *The hypothesis of Inverse function theorem is only sufficient but not neccessary. For example* $f(x) = x^3$. $f'(0) = 0$ *and the function has* C^1 *inverse in around* $x = 0$.