Chapter 4
Parabolic Equations

4.1 Physical models

Let $u(x,t)$ be the heat of material at the point x and at time t in a uniform cross section object. Let $q(x,t)$ be the heat flux and $f(x,t)$ is the internal generated sources. Then from the equation of continuity we get

$$\int_{x_1}^{x_2}[u(x,t_2)-u(x,t_1)]\,dx=\int_{x_1}^{x_2}\int_{t_1}^{t_2}f(x,t)dtdx+\int_{t_1}^{t_2}[q(x_1,t)-q(x_2,t)]\,dt$$

This is equivalently,

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial u}{\partial t}(x,t)dxdt=\int_{x_1}^{x_2}\int_{t_1}^{t_2}f(x,t)dxdt-\int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial q}{\partial x}(x,t)dxdt$$

Therefore,

$$\frac{\partial u}{\partial t}=f(x,t)-\frac{\partial q}{\partial x}$$

Now using the Fourier Law, we assume that $q=-k\frac{\partial u}{\partial x}$, where k is viscosity constant that depends on the material. In case of $k=1$, we get the equation

$$\frac{\partial u}{\partial t}=f(x,t)+\frac{\partial^2 u}{\partial x^2},\ x\in\mathbb{R},t>0.$$

In higher dimensions, $x\in\mathbb{R}^n$, we get

$$\frac{\partial u}{\partial t}=f(x,t)+\Delta u,\ x\in\mathbb{R}^n,\ t>0.$$

4.2 Fundamental solution

We consider the problem

$$u_t=\Delta u,\ t>0,\mathbb{R}^n. \tag{2.1}$$

We note that if $u(x,t)$ solves this equation, then $u(\lambda^2 t,\lambda 0x)$ also solves for $\lambda\in\mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t},(r=|x|)$ is important for heat equation. So we can search for solution $u(x,t)$ of the form

$$u(x,t) = v\left(\frac{r^2}{t}\right),\ r = |x|,\ t > 0,\ x \in \mathbb{R}^n$$

So we assume that

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right),\ x \in \mathbb{R}^n,\ t > 0,$$

where α, β will be found. Taking $y = \frac{x}{t^\beta}$, we get

$$u_t = -\left(\frac{\alpha}{t^{\alpha+1}} v(y) + \frac{\beta}{t^{\alpha+1}} \nabla v \cdot y\right)$$
$$\Delta u = \frac{1}{t^{\alpha+2\beta}} \Delta v$$

Substituting this in the equation (2.1) and taking $\beta = \frac{1}{2}$, we get

$$u_t - \Delta u = \frac{1}{t^{\alpha+1}}\left(\alpha v(y) + \frac{1}{2}\nabla v \cdot y + \Delta v\right) = 0 \tag{2.2}$$

We simplify this further by taking $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Then

$$\frac{\partial v}{\partial y_i} = w'(r)\frac{y_i}{|y|} = w'(r)\frac{y_i}{|y|} \implies \nabla v \cdot y = w'(r)r$$
$$\Delta v = w'' + \frac{n-1}{r} w'$$

Therefore from (2.1), we get

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$$

Now setting $\alpha = \frac{n}{2}$ we can write the above equation in the exact form as

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0$$

Upon integration, we get $r^{n-1} w'(r) + \frac{1}{2} r^n w = a$. Assuming $\lim_{r\to\infty} w(r), w'(r) = 0$, we get $w(r) = b e^{\frac{-r^2}{4}}$, where b is a constant of integration. Therefore we have the

$$u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}$$

Definition 4.2.1 *Fundamental solution: The function*

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\dfrac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$$

is called fundamental solution. The choice of $b = \frac{1}{(4\pi)^{n/2}}$ is due to the following

Lemma 4.2.1 *The fundamental solution $\Phi(x,t)$ satisfies*

1. $\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1,$

2. $\lim_{t\to 0^+} \Phi(x,t) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$
3. $\Phi_t - \Delta_x \Phi = 0$, $x \in \mathbb{R}^n$, $t > 0$.

Proof. 1. Taking the transformation $z = \frac{x}{2\sqrt{t}}$ we get

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} dz = 1.$$

2. follows directly from the definition.
3. Direct calculation from the definition. □

Lemma 4.2.2 *$\Phi(x,t) \to \delta$ in $\mathscr{D}'(\mathbb{R}^n)$. i.e.,*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} \Phi(x,t)\psi(x)dx = \psi(0), \textit{ for all } \psi \in \mathscr{D}(\mathbb{R}^n). \tag{2.3}$$

Proof. Let $\psi \in \mathscr{D}(\mathbb{R}^n)$. We notice that

$$\psi(0) = \int_{\mathbb{R}^n} \Phi(x,t)\psi(x)dx + \int_{\mathbb{R}^n} \Phi(x,t)(\psi(0) - \psi(x))dx$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x| < \delta \implies |\psi(x) - \psi(0)| < \varepsilon$$

From (1) of above Lemma, we get

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \Phi(x,t)(\psi(0) - \psi(x))dx \right| &\leq \int_{\mathbb{R}^n} \Phi(x,t)\, |\psi(0) - \psi(x)|\, dx \\
&= \int_{B_\delta(0)} + \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x,t)\, |\psi(0) - \psi(x)|\, dx \\
&:= I_\varepsilon + J_\varepsilon \\
&\leq \varepsilon \int_{\mathbb{R}^n} \Phi(x,t)dx + J_\varepsilon = \varepsilon + J_\varepsilon
\end{aligned}$$

$|J_\varepsilon| \leq 2\|\psi\|_\infty \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x,t)dx \leq \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{t}} r^{n-1} dr \longrightarrow 0$ as $t \to 0^+$. □

Theorem 4.2.1 *Φ is a fundamental solution of $(\partial_t - \Delta)$ in $\mathbb{R}^{n+1}$. i.e.,*

$$(\partial_t - \Delta_x)\Phi(x,t) = \delta(x,t) \textit{ in } \mathscr{D}'(\mathbb{R}^{n+1}),$$

where $\delta(x,t)$ is the Dirac delta distribution at $(0,0)$.

Proof. Let $\phi \in \mathscr{D}(\mathbb{R}^{n+1})$. Then define the cut off functions

$$\Phi_\varepsilon(x,t) = \begin{cases} \Phi(x,t), & t > \varepsilon \\ 0, & t \leq \varepsilon \end{cases}$$

Then $\Phi_\varepsilon \to \Phi$ in $\mathscr{D}'(\mathbb{R}^{n+1})$. So it enough to show $(\partial_t - \Delta_x)\Phi_\varepsilon \to \delta$ in $\mathscr{D}'(\mathbb{R}^{n+1})$. Indeed, for $\phi \in \mathscr{D}(\mathbb{R}^{n+1})$, using integration by parts and the compact support of ϕ,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(-\partial_t - \Delta_x)\phi &= \int_\varepsilon^\infty \int_{\mathbb{R}^{n+1}} \Phi(-\partial_t - \Delta)\phi dxdt \\ &= \int_{\mathbb{R}^{n+1}} [(-\partial_t - \Delta_x)\Phi_\varepsilon]\phi + \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\phi(x,e)dx \\ &= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\phi(x,\varepsilon)dx\end{aligned}$$

But then

$$\begin{aligned}\left|\phi(0,0) - \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\phi(x,\varepsilon)dx\right| &= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)(\phi(x,\varepsilon) - \phi(0,0))dx \\ &= \int_{B_\delta(0)} + \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x,\varepsilon)(\phi(x,\varepsilon) - \phi(0,0))dx \\ &= I + J\end{aligned}$$

For any $\eta > 0$ there exists $\delta > 0$ such that $\sqrt{x^2+\varepsilon^2} < \delta$ implies $|\phi(x,\varepsilon) - \phi(0,0)| < \eta$. Therefore, $|I| < \eta$. To estimate J we note that

$$|J| \leq 2\|\phi\|_\infty \int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x,\varepsilon)dx \leq \frac{C}{\varepsilon^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{\varepsilon}} r^{n-1} dr \longrightarrow 0 \text{ as } \varepsilon \to 0^+.$$

Now δ and ε can be chosen to be small so that $I+J$ is as small as possible. Hence the theorem. □

4.3 Cauchy Problem

Consider the problem: Given $g(x)$, find $u(x,t)$ satisfying

$$(CP) \quad \begin{cases} u_t - \Delta u = 0, \; x \in \mathbb{R}^n, \; t > 0 \\ u(x,0) = g(x), \; x \in \mathbb{R}^n \end{cases}$$

We have the following theorem

Theorem 4.3.1 *Suppose $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$u(x,t) = (\Phi * g)(x) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy$$

is a solution of the Cauchy Problem (CP)

Proof. It is easy to see that Φ is differentiable for any $(x,t) \in (\mathbb{R}^n \times [\delta,\infty))$ and all derivatives are bounded and integrable. Therefore we can take the derivatives inside the integral sign to get

$$u_t - \Delta u = \int_{\mathbb{R}^n} (\Phi_t - \Delta_x \Phi)(x-y,t)g(y)dy = 0.$$

It remians to show that $\lim_{t\to 0^+,\, x \to x_0} u(x,t) = g(x_0)$. Using the properties of Φ, and the continuity of g, $\varepsilon > 0$, there exists 4̣ > 0 such that

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon.$$

Then it is easy to see the following

$$\begin{aligned}
|g(x_0) - u(x,t)| &\leq \left| \int_{\mathbb{R}^n} \Psi(x-y,t)(g(x_0)-g(y))dx \right| \\
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t)\,|g(x_0)-g(y)|\,dx \\
&= \int_{B_\delta(x_0)} + \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \Phi(x-y,t)\,|g(x_0)-g(y)|\,dx \\
&:= I_\varepsilon + J_\varepsilon \\
&\leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)dy + J_\varepsilon = \varepsilon + J_\varepsilon
\end{aligned}$$

To estimate J_ε, note that in this integral $|y-x_0| \geq \delta$ and as $x \to x_0$, we may assume that $|x-x_0| < \delta/2$. Hence we get

$$|y-x_0| \leq |y-x| + |x-x^0| \leq |y-x| + \frac{\delta}{2} \leq |y-x| + \frac{1}{2}|y-x_0|$$

Therefore, we get $|y-x| \geq \frac{1}{2}|y-x_0|$. Using this we estimate J_ε as

$$|J_\varepsilon| \leq 2\|g\|_\infty \int_{\mathbb{R}^n \setminus B_\delta(x)} \Phi(x-y,t)dx \leq \frac{C}{t^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{t}} r^{n-1} dr \longrightarrow 0 \text{ as } t \to 0^+.$$

Hence, if $|x-x_0| < \frac{\delta}{2}$ and $t > 0$ small, we get $|u(x,t) - g(x_0)| < 2\varepsilon$. □

The following stability estimate follows from the properties of Φ

Corollary 4.3.1 *If g is continuous and bounded, then*

$$\|u(.,t)\|_\infty \leq \|g\|_\infty, \textit{ for all } t > 0.$$

Next we consider the following nonhomogeneous problem:

$$(CPN) \qquad \begin{cases} u_t - \Delta u = f(x,t),\ x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = 0,\ x \in \mathbb{R}^n. \end{cases}$$

Duhamel's Principle: This is a general principle of getting solutions of nonhomogeneous equation using the solutions of homogeneous problems. To understand the principle, let us recall the ODE case. The nonhomogeneous IVP:

$$y' + ay = b(t),\ y(t_0) = 0$$

has solution

$$y(t) = \int_0^t b(s)e^{-a(t-s)}ds \tag{3.4}$$

In otherwords,

$$y(t) = \int_0^t x(t-s)b(s)ds$$

where $x(t)$ satisfies the homogeneous problem $x' + ax = 0$. Moreover $x(s,t) = b(s)e^{-a(t-s)}$ satisfies the IVP:

$$x' + ax = 0,\ x(s) = b(s) \tag{3.5}$$

So the formula in (3.4) may be written

$$y(t) = \int_0^t x(t,s)b(s)ds$$

Now going back to the heat equation (CPN), we write the homogeneous problem similar to (3.5) as: $U(x,t,s)$ satisfies

$$\begin{cases} U_t(x,t,s) - \Delta_x U(x,t,s) = 0, \ t > s, \ x \in \mathbb{R}^n, \\ U(x,t,s) = f(x,s) \text{ on } \{t = s\} \end{cases} \tag{3.6}$$

Then $U(x,t,s)$ may be written as

$$U(x,t,s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dy.$$

Theorem 4.3.2 *If $f \in C^2(\mathbb{R}^n \times \mathbb{R}^+)$, $f(x,t)$ and all its second order partial derivatives are continuous and bounded. Then the function $u(x,t)$ defined as*

$$u(x,t) = \int_0^t U(x,t,s) ds$$

solves the problem (CPN).

Proof. From the definition of U, we have

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dyds$$

taking the transformation $x - y \mapsto y$ and $t - s \mapsto s$ we get

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) dyds$$

Then using Newton-Liebnitz formula,

$$\begin{aligned} u_t &= \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial}{\partial t} f(x-y,t-s) dyds \\ \Delta u &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \Delta_x f(x-y,t-s) dyds \end{aligned}$$

Therefore,

$$\begin{aligned} u_t - \Delta u &= \int_0^\varepsilon \int_{\mathbb{R}^n} + \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s) dyds + \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,0) dy \\ &= J_\varepsilon + I_\varepsilon + K \end{aligned} \tag{3.7}$$

Since $f \in C_c^2(\mathbb{R}^n)$, we have the estimate

$$|J_\varepsilon| \le C\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) dy \to 0, \text{ as } \varepsilon \to 0$$

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s) dyds \\ &= -\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \partial_s f(x-y,t-s) dyds - \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \Delta_y f(x-y,t-s) dyds \end{aligned}$$

Integration by parts on the first term, yields

$$
\begin{aligned}
-\int_{\varepsilon}^{t}\int_{\mathbb{R}^n}\Phi(y,s)\partial_s f(x-y,t-s)dyds &= \int_{\varepsilon}^{t}\int_{\mathbb{R}^n}\frac{\partial}{\partial s}\Phi(y,s)f(x-y,t-s)dyds \\
&\quad -\int_{\mathbb{R}^n}\Phi(y,s)f(x-y,t-s)\Big|_{s=\varepsilon}^{t} \\
&= \int_{\varepsilon}^{t}\int_{\mathbb{R}^n}\frac{\partial}{\partial s}\Phi(y,s)f(x-y,t-s)dyds \\
&\quad -\int_{\mathbb{R}^n}\Phi(y,t)f(x-y,0)dy+\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)dy
\end{aligned}
$$

Again integration by parts on second term and using the fact that f has compact support we get

$$
-\int_{\varepsilon}^{t}\int_{\mathbb{R}^n}\Phi(y,s)\Delta_y f(x-y,t-s)dyds = -\int_{\varepsilon}^{t}\Delta_y\Phi(y,s)f(x-y,t-s)dyds
$$

Putting these things back in I_ε we get

$$
I_\varepsilon = K+\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)dy+\int_{\varepsilon}^{t}\int_{\mathbb{R}^n}(\Phi_t-\Delta_y\Phi)f(x-y,t-s)dyds
$$

The last term is equal to zero as $t=0$ is not in the domain of integration and Φ satisfies the heat equation for all $t>0$. So from (3.7), we get

$$
u_t-\Delta u = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)dy
$$

As in the previous theorems, noting that for any $\eta>0$, there exists a $\delta>0$ such that

$$
|y|+\varepsilon<\delta \implies |f(x-y,t-\varepsilon)-f(x,t)|<\eta
$$

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n}\Phi(y,\varepsilon)f(x-y,t-\varepsilon)dy-f(x,t)\right| &\le \int_{\mathbb{R}^n}\Phi(y,\varepsilon)\,|f(x-y,t-\varepsilon)-f(x,t)|\,dy \\
&= \int_{B_\delta(0)}+\int_{\mathbb{R}^n\setminus B_\delta(0)}\Phi(y,\varepsilon)\,|f(x-y,t-\varepsilon)-f(x,t)|\,dy \\
&\le \eta\int_{\mathbb{R}^n}\Phi(y,\varepsilon)dy+2\|f|_\infty\int_{\mathbb{R}^n\setminus B_\delta(0)}\Phi(y,\varepsilon)dy \\
&\le \eta+\frac{C}{\varepsilon^{n/2}}\int_{\delta}^{\infty}e^{-\frac{r^2}{\varepsilon}}r^{n-1}dr \longrightarrow 0 \text{ as } \eta,\varepsilon\to 0^+
\end{aligned}
$$

Hence the proof of the theorem. □

4.4 Maximum Principles

The heat equation also satisfies the maximum principles. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $T>0$. Define $\Omega_T=\Omega\times(0,T)$. We define the parabolic boundary

$$
\Gamma=\{(x,t)\in\overline{\Omega_T}:\ x\in\partial\Omega \text{ or } t=0\}
$$

Theorem 4.4.1 *Let $u\in C^2(\Omega_T)\cap C(\overline{\Omega_T})$ and all its second order partial derivatives are continuous. Suppose u satisfy*

$$u_t \le \Delta u \quad in\ \Omega_T.$$

Then u achieves its maximum on the parabolic boundary of Ω_T. i.e.,

$$\max_{\overline{\Omega_T}} u(x,t) = \max_{\Gamma} u(x,t)$$

Proof. As in the elliptic case, we proceed in two steps.

1. case 1: $u_t < \Delta u$ in Ω_T.
   For $0 < \tau < T$, consider

   $$\Omega_\tau = \Omega \times (0,\tau),\ \ \Gamma_\tau = \{(x,t) \in \overline{\Omega_\tau},\ x \in \partial\Omega\ or\ t = 0\}$$

   If the maximum of u is on $\overline{\Omega_\tau}$ occurs at $x \in \Omega$ and $t = \tau$, then $u_t(x,\tau) \ge 0$ and $\Delta u(x,\tau) \le 0$, violating our assumption. Similarly, u cannot attain an interior maximum on Ω_τ. Hence the result holds on Ω_τ:

   $$\max_{\overline{\Omega_\tau}} u(x,t) = \max_{\Gamma_\tau} u(x,t) \le \max_{\Gamma} u(x,t)$$

   Now by continuity of u, $\max_{\overline{\Omega_T}} u = \lim_{\tau \to T} \max_{\overline{U_\tau}} u$. This proves the theorem in case 1.
2. case 2: $u_t \le \Delta u$ in Ω_T.
   Let $v = u - kt$ for some $k > 0$. Notice that $v \le u$ on $\overline{\Omega_T}$ and $\Delta v - v_t = \Delta u - u_t + k > 0$ in Ω_T. Thus by case 1,

   $$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} (v + kt) = \max_{\overline{\Omega_T}} v + kT = \max_{\Gamma} v + kT \le \max_{\Gamma} u + kT$$

   letting $k \to 0$ establishes the theorem. □

As a corollary we have the following uniqueness result

Corollary 4.4.1 *The initial boundary value problem*

$$u_t - \Delta u = f(x,t)\ in\ \Omega_T$$
$$u(\partial\Omega,t) = g(x,t),\ u(x,0) = h(x),\ x \in \Omega$$

has at most one solution.

Proof. Suppose u_1 and u_2 are two solutions. Then $u = u_1 - u_2$ satisfies the problem

$$u_t - \Delta u = 0 \text{ in } \Omega_T$$
$$u(\partial\Omega,t) = 0,\ u(x,0) = 0,\ x \in \Omega$$

Since the maximum of u on the parabolic boundary is 0, by above theorem, $u \equiv 0$. □

Next we prove the following **Maximum principle for Cauchy problem**

Theorem 4.4.2 *Suppose $u(x,t)$ and all its second order Partial derivatives are continuous in $\mathbb{R}^n \times (0,T)$, $u(x,t)$ is continous in $\mathbb{R}^n \times (0,T]$ and*

$$u(x,t) \le Ae^{a|x|^2},\ x \in \mathbb{R}^n,\ 0 \le t \le T$$

for some $A, a > 0$. If u is a solution of the Cauchy problem

$$u_t = \Delta u,\ x \in \mathbb{R}^n, t > 0$$
$$u(x,0) = g(x),\ x \in \mathbb{R}^n$$

Then

$$\sup_{\mathbb{R}^n \times [0,T]} u(x,t) = \sup_{\mathbb{R}^n} g(x).$$

Proof. Let us first assume that $4aT < 1$. Then there exists $\varepsilon > 0$ such that $4a(T+\varepsilon) < 1$. Fix $y \in \mathbb{R}^n, \mu > 0$ and define

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \quad x \in \mathbb{R}^n, t > 0.$$

Then $v_t - \Delta v = 0$ in $\mathbb{R}^n \times (0,T]$. For $r > 0$, consider $\Omega_T = B_r(y) \times (0,T]$. Then by previous maximum principle,

$$\max_{\overline{\Omega_T}} v(x,t) = \max_{\Gamma_T} v(x,t). \tag{4.8}$$

Now note that

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \leq u(x,0) = g(x).$$

Also if $y \in \partial B_r(x), 0 \leq t \leq T$,

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}} \\ &\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}} \end{aligned}$$

Now $4a(T+\varepsilon) < 1$ implies $\frac{1}{4(T+\varepsilon)} = a + \gamma$ for some $\gamma > 0$. Therefore,

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2}$$

Taking $y = 0$ and if r is taken large enough, we get

$$v(x,t) \leq Ae^{ar^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{\mathbb{R}^n} g(x)$$

Therefore, by (4.8)

$$v(y,t) \leq v(x,t) \leq \sup g, \text{ for all } y \in \mathbb{R}^n, 0 \leq t \leq T.$$

Now taking supremum over $|x| = r$ we get

$$\sup_{\overline{B_r} \times [0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g(x)$$

Now taking $r \to \infty$ and $\mu \to 0$ we get the result. In case $4aT \geq 1$. Then apply this result for $[0,T_1], [T_1,T_2]$ with $T_1 = \frac{1}{8a}$. $\square$

Corollary 4.4.2 *Uniqueness: The Cauchy problem*

$$u_t = \Delta u + f(x,t),\ x \in \mathbb{R}^n, t > 0$$
$$u(x,0) = g(x),\ x \in \mathbb{R}^n$$

has at most one solution which has the growth $u(x,t) \leq Ae^{a|x|^2}$.

Proof. If u_1 and u_2 are two solutions. Then consider the function $u = u_1 - u_2$. The Linearity of $\partial_t - \Delta$ suggests that u satisfies

$$u_t = \Delta u,\ x \in \mathbb{R}^n, t > 0$$
$$u(x,0) = 0,\ x \in \mathbb{R}^n$$

Then by the maximum principle, $u \equiv 0$. □

Now recall that in case of harmonic functions all the distributional solutions of $\Delta u = 0$ are C^∞ (Weyl theorem). In case of parabloic, we have the following Regularity result:

Theorem 4.4.3 *If $u(x,t)$ satisfies $u_t = \Delta u$ in the sense of distributions in U. Then $u \in C^\infty(U)$.*

Proof. Let $\xi = (x_0, t_0) \in U$ and $\varepsilon > 0$. Consider $\phi \in \mathscr{D}(B_{4\varepsilon}) \subset U$ such that $\phi \equiv 1$ in $B_{3\varepsilon}(\xi)$. Consider $w = \phi u$ and $v = \partial_t w - \Delta w$, then v is a distribution in U with support in $B_{4\varepsilon}$ and $v = 0$ in $B_{3\varepsilon}$. Then we know from theory of Distributions that

$$w = \Phi * v \text{ in } \mathscr{D}'.$$

Now we claim that w is smooth around ξ. Choose $\psi \in C^\infty(\mathbb{R}^{n+1})$ with $\psi = 0$ in $B_\varepsilon(0)$ and $\psi = 1$ in $\mathbb{R}^{n+1} \backslash B_{2\varepsilon}(0)$. Then ψ vanishes in a neighbouhood of the singularity of $\Phi(y,s)$. So $\psi\Phi$ is a smooth function.

Now for $(x,t) \in B_\varepsilon(\xi)$ and $(y,s) \in supp(v) \subset B_{4\varepsilon}(\xi) \cap (\mathbb{R}^{n+1} \backslash B_{3\varepsilon}(\xi))$, we have

$$|(x,t) - (y,s)| \geq 2\varepsilon \implies \psi(x-y, t-s) = 1.$$

Therefore,

$$(\psi\Phi) * v = \Phi * v = w,\ (x,t) \in B_\varepsilon(\xi)$$

which implies that w is smooth on $B_\varepsilon(\xi)$. But $w = \phi u$ and $\phi = 1$ in B_ε. Therefore u is smooth.

4.5 Problems

1. Show that $\int_{\mathbb{R}^n \backslash B_\delta(0)} \Phi(x,t)dx \to 0$ as $t \to 0^+$.
2. $\Phi_\varepsilon \to \Phi$ in $\mathscr{D}'(\mathbb{R}^{n+1})$
3. Suppose $U = \Omega \times (0,T)$ and $u \in C^3(U) \cap C(\overline{U})$ satisfies $u_t \leq \Delta u + cu$ in U, where $c \leq 0$ is a constant. If $u \geq 0$, show that
$$\max_{(x,t) \in \overline{U}} u(x,t) = \max_{(x,t) \in \Gamma} u(x,t)$$
where $\Gamma = \{(x,t) \in \overline{U} : x \in \partial\Omega \ or\ t = 0\}$.