Chapter 2
Higher order Equations

This theorem explores the method of solving PDEs using power series expansion, with the underlying assumption that any realistic solution must have a convergent power series expansion(that is real analytic) in some neighbourhood of the initial point/surface. However, analyticity is not an appropriate requirement as many real world problems have non-real analytic solutions and hence one needs also study the existence and uniqueness of non-analytic solutions. Yet this is a powerful tool to understand the local solutions.
The theorem of Cauchy-Kovalevski asserts the local existence of solutions to system of Initial value problems of partial differential equations with initial conditions prescribed on a non-characteristic hyper surface. There is a severe restriction that the coefficients in the equations, the initial data and the surface need to be real analytic.
The following questions are naturally asked.

- Does there exists C^∞ solution when coefficients and data is replaced by differentiability? In particular, does a PDE with C^∞ data in general has unique C^∞ solution?
- Can there be non-analytic solutions for Cauchy problem with real analytic data?. In other words, does there exists two or more solutions with only one of them being real analytic.

The answer to the first question is no. For the second question, when the equation is linear, Holmgren's theorem guarantees that the answer is no. When the equation is nonlinear, the problem is still unsettled.

We will see that the theorem only asserts existence. It does not say anything about well posedness. That is a small change in the initial condition may lead to large change in the series solution.

2.1 Cauchy Problem: Higher order equations

Let $(x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ and $x = (x', x_n)$. Consider the probem

$$\frac{\partial^m u}{\partial x_n^m} = G(x, D^\alpha u),\ x = (x', x_n), x_n > 0 \tag{1.1}$$

with initial data

$$u = \frac{\partial u}{\partial x_n} = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0 \text{ on } \{x = (x', x_n) : x_n = 0,\}$$

where $D^\alpha u$ is the vector of all partial derivatives up to order m, except $\dfrac{\partial}{\partial x_n^m}$.

A general non characteristic initial value problem for a system of quasilinear partial differential equation can always be reduced to the following system of first order equations:

$$\frac{\partial u_i}{\partial x_n} = \sum_{k=1}^{n-1}\sum_{j}^{N} a_{ij}^k(P)\frac{\partial u_j}{\partial x_k} + b_i(P), \;\; i = 1,2,...,N \tag{1.2}$$

with initial conditions/ Cauchy data: $u_i = 0$ on $\{x = (x', x_n),\, x_n = 0\}$ for $i = 1,2,...,N$.
Here we may assume that the coefficients $a_{i,j}^k$ is independent of x_n. Indeed, we can define a new variable $u_{N+1} = x_n$ and add one more equation $\frac{\partial u_{N+1}}{\partial x_n} = 1$ to the system.

The problem in (1.2) can be transformed as Cauchy problem for system. In fact, one can define $u_1 = u, u_2 = \frac{\partial u_1}{\partial x_n} = ...u_N = \frac{\partial^m u}{\partial x_n^m} = G(x, u_1, u_2, ...u_N)$. We describe the procedure below.
For simplicity, consider the following second order equation, for $a, b, c, d, e \in \mathbb{R}$

$$a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial u}{\partial t} + e\frac{\partial u}{\partial x} + f = 0 \tag{1.3}$$

Consider the transformation

$$u_1 = u,\; u_2 = u_t,\; u_3 = u_x.$$

Then

$$\frac{\partial u_1}{\partial x} = u_3,\; \frac{\partial u_2}{\partial x} = \frac{\partial u_3}{\partial x}\left(= \frac{\partial^2 u}{\partial t \partial x}\right)$$

and (1.3) imply

$$a\frac{\partial u_2}{\partial t} + b\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial t}\right) + c\frac{\partial u_3}{\partial x} + du_2 + eu_3 + f = 0.$$

By considering the vector $U = (u_1, u_2, u_3)^T$, we get

$$A\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = CU + F$$

where A, B and C are 3×3 matrices and F is a vector.

Example 2.1. $u_{tt} - c^2 u_{xx} = 0$.
Here $a = 1, b - 0, c = c^2, d = e = f = 0$, setting $u_1 = u, u_2 = u_t, u_3 = u_x$, we get

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= u_3 \\ \frac{\partial u_2}{\partial x} &= \tfrac{\partial u_3}{\partial t} \\ \frac{\partial u_2}{\partial t} - c^2\frac{\partial u_3}{\partial x} &= 0 \end{aligned}$$

Then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}\begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_3}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

i.e.,

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial t} + CU = 0$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As A is nonsingular, multiplying by A^{-1} we get

$$\frac{\partial U}{\partial x} + \bar{B}\frac{\partial U}{\partial t} + \bar{C}U = 0.$$

The Cauchy Kovalevski theorem states:

Theorem 2.1.1 *Assume that the functions a_{ij}^k and $b_i(i; j = 1;2;:::;N; k = 1;2;:::;n)$ are real analytic in a neighbourhood of the origin in R^{n+N-1}. Then the Cauchy problem in (1.1) admits a unique real analytic solution in a neighbourhood of the origin.*

Example 2.2. Consider the following IVP:

$$u' - u^3 = 0, \ u(0) = 1$$

We will try to construct the unknown function u by computing its derivatives at the initial time 0. Then formally we have

$$u(t) = u(0) + u'(0)t + \frac{1}{2}u''(0)t^2 + ...$$

This formal relation becomes rigorous when the power series on the right hand side converges in some neighbourhood of 0.
The first two terms in the series are already known. It turns out that we can compute the remaining terms recursively using the given initial values and the differential equation:

$$\begin{aligned} u'(0) &= u(0)^3 = 1; \\ u''(0) &= 3u(0)u'(0) = 3; \\ u^3(0) &= 3u(0)u''(0) + 6u(0)u'(0)^2 = 15; \end{aligned}$$

Then it is clear that we only need to show the convergence of the series

$$1 + t + \frac{3}{2}t^2 + \frac{15}{3!}t^4 + ...$$

However it is not clear how to directly bound these coefficients.

Example 2.3. Consider the Cauchy Problem

$$u_{xx} + uu_{yy} - u_y = u^2, \ \ u(x,0) = 1, \ \ u_y(x,0) = x$$

In the normal form

$$u_{yy} = u + \frac{u_y - u_{xx}}{u}.$$

Now it is easy to see that all partial derivatives of u can be obtained. From the given initial condition we have

$$u_x(x,0) = 0,\ u_{xy}(x,0) = 1$$
$$u_y(x,0) = x,\ u_{xx}(x,0) = 0$$
$$u_{yy}(x,0) = u(x,0) + \frac{u_y(x,0) - u_{xx}(x,0)}{u(x,0)} = 1 + x$$

Similarly we can obtain partial derivatives of all orders. Now we can write the Taylor series of u

$$u(x,y) = u(x,0) + yu_y(x,0) + \frac{1}{2}y^2 u_{yy}(x,0) + ... = 1 + y(x) + \frac{y^2}{2}(1+x) + ...$$

Does this series converge?

Cauchy problem from Complex Analysis view

Consider the following Cauchy problem for the linear PDE:

$$w_y - iw_x = 0,\ w(x,0) = g(x)$$

where $w = u + iv$ is complex valued function. By identifying $z = x + iy$, we may also consider w as a function of complex variable z. Now if $w \in C^1$ satisfies this equation, by Cauchy-Riemann equations we know that w is a analytic function and has an absolutely convergent Taylor series about any point. Then we find that $w(x + i0) = g(x)$ has absolutely convergent Taylor series. Hence g is real analytic. **That is, the problem has a solution only when real and imaginary parts of g are real analytic**. Conversely, if $g(x) = \sum a_n(x - x_0)^n$ is absolutely convergent for $|x - x_0| < R$, then we can define $w(z) = \sum a_n(z - x_0)^n$, which is absolutely convergent for $|z - x_0| < R$. Therefore, w is an analytic function and hence w must satisfy Cauchy-Riemann equations: $w_y = iw_x$. Therefore, this problem has C^1 solution if and only if g is real analytic.

Now let us try to construct a Taylor series for the two-variable function $w(x,y)$ satisfying

$$w_y = iw_x,\ \ w(x,0) = g(x)$$

The Taylor series of $w(x,y)$ around $(0,0)$ is

$$u(x,y) = \sum \frac{\partial_y^j \partial_x^k w(0,0)}{j!k!} y^j x^k,$$

and then show its convergence. Note that the uniqueness part is trivial.

$$w(0,0) = g(0)$$
$$\partial_x w(0,0) = g'(0)$$
$$\partial_y w(0,0) = i\partial_x w(0,0) = ig'(0)$$
$$\partial_x^2 w(0,0) = g''(0)$$
$$\partial_y \partial_x w(0,0) = i\partial_x^2 w(0,0) = ig''(0)$$
$$\partial_y^2 w(0,0) = i\partial_y \partial_x w(0,0) = i^2 g''(0)$$

Now using the equation, we can compute all derivatives as

$$\partial_y^j \partial_x^k w(0,0) = i\partial_y^{j-1} \partial_x^{k+1} w(0,0) = ... = i^j \partial_x^{k+j} u(0,0) = i^j g^{(j+k)}(0).$$

Thus we obtain the Taylor series

$$\sum_{j,k=0}^{\infty} \frac{i^j g^{(j+k)}(0)}{j!k!} y^j x^k.$$

Now let us assume that

$$g(x) = \sum_{l=0}^{\infty} \frac{g^l(0)}{l!} x^l$$

converges for all $|x| \leq R$. This implies the existence of a constant C such that

$$|g^{(l)}(0)| \leq C \frac{l!}{R^l}.$$

Setting $l = j + k$ we see that the series of w is dominated by the series

$$C \sum_{j,k=0}^{\infty} \frac{(j+k)!}{j!k!} \left|\frac{y}{R}\right|^j \left|\frac{x}{R}\right|^k$$

which can be re-ordered into

$$C \sum_{m=0}^{\infty} \sum_{j+k=m} \frac{m!}{j!(m-j)!} \left|\frac{y}{R}\right|^j \left|\frac{x}{R}\right|^{m-j} = C \sum_{m=0}^{\infty} \left(\frac{|y|+|x|}{R}\right)^m.$$

It is clear that the convergence holds whenever $|y| + |x| < R$.
For equations which are not linear, it can be hard to obtain formulas for the coefficients and estimate their sizes. The method of majorants in an indirect way to bound the coefficients. Here we describe this first for ODE

2.2 Cauchy Kovalevski theorem for ODE

Theorem 2.2.1 *Let $f : (-1,1) \to \mathbb{R}$ be real analytic in some neighbourhood of* 0 *and $u(x)$ is the unique solution of semilinear ODE*

$$\frac{du}{dx} = f(u(x)),\ u(0) = 0.$$

Then u is also real analytic in a neighbourhood of 0*.*

Proof. There are several simple proofs. Here we describe a proof which may be generalized to Cauchy problem of PDE.
Suppose u is C^∞, then by repeated differentiation we see that

$$\begin{aligned}
\frac{d^2u}{dx^2} &= f'(u)\frac{du}{dx} = f'(u)f(u) \\
\frac{d^3u}{dx^3} &= f''(u)f^2(u) + (f'(u))^2 f(u) \\
&. \\
&. \\
&. \\
\frac{d^nu}{dx^n} &= p_n(f(u), f'(u), f''(u), \dots, f^{n-1}(u))
\end{aligned}$$

where p_n is a polynomial in n variables with all non-negative integer coefficients. It is easy to see that

$$p_2(x,y) = xy,\ p_3(x,y,z) = x^2z + xy^2$$

and so on. Note that these polynomials do not depend on $f(t)$. Also,

$$\frac{d^n u}{dx^n}(0) = p_n(f(0), f'(0), f''(0), ..., f^{n-1}(0))$$

Therefore,

$$\begin{aligned} |\frac{d^n u}{dx^n}(0)| &= |p_n(f(0), f'(0), f''(0), ..., f^{n-1}(0))| \\ &\le p_n(|f(0)|, |f'(0)|, |f''(0)|, ..., |f^{n-1}(0)|). \end{aligned} \tag{2.4}$$

Now suppose there exists a real analytic function $g(t)$ such that $|f^k(0)| \le g^k(0)$ for all k and $\frac{dv}{dt} = g(v(t)),\ v(0) = 0$ has real analytic solution.
Then by above observation as in (2.4),

$$\frac{d^n v}{dx^n}(0) = p_n\left(g(0), g'(0), g''(0), ..., g^{n-1}(0)\right) \tag{2.5}$$

and

$$p_n\left(|f(0)|, |f'(0)|, |f''(0)|, ..., |f^{n-1}(0)|\right) \le p_n\left(g(0), g'(0), g''(0), ..., g^{n-1}(0)\right) \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$v^n(0) \ge |u^n(0)| \text{ for all } n.$$

Since v is real analytic with radius of convergence ρ(say), then

$$\sum_{n=0}^{\infty} \frac{1}{n!}|u^n(0)|\rho^n \le \sum_{n=0}^{\infty} \frac{1}{n!} v^n(0)\rho^n < \infty$$

and hence the following function is well defined for $|x| < r$,

$$w(x) := \sum_{n=0}^{\infty} \frac{1}{n!} p_n\left(f(0), f'(0), f''(0), ..., f^{n-1}(0)\right) x^n = \sum_{n=0}^{\infty} \frac{1}{n!} u^{(n)}(0) x^n. \tag{2.7}$$

Now since expression in (2.7) is real analytic, it is easy to check that $w(x)$ is a solution by comparing the derivatives at 0.
To complete the proof, we show the existence of v and g. To see this, we know that $\sum_{n=0}^{\infty} \frac{1}{n!} f^n(0)\rho^n$ converges. Let

$$C = \max_n |\frac{1}{n!} f^n(0)\rho^n|.$$

Then $\max_n |\frac{1}{n!} f^n(0)| \le C\rho^{-n}$. Now we define

$$g(t) = \sum_{n=0}^{\infty} C\left(\frac{t}{\rho}\right)^n = C\frac{1}{1-\frac{t}{\rho}} = \frac{C\rho}{\rho - t}$$

Clearly, $|f^n(0)| \leq g^n(0)$. Consider the problem

$$\frac{dv}{dx} = \frac{C\rho}{\rho - v},\ v(0) = 0$$

i.e.,

$$vv' - \rho v' + C\rho = 0$$

In other words, $\frac{1}{2}d(v^2) - d(\rho v) + C\rho = 0$. Hence $v(x) = \rho - \sqrt{\rho^2 - 2C\rho x}$. Now it is easy to see that v is real analytic for $|x| < \frac{\rho}{C}$.

2.2.1 Multi Index of Laurent- Schwarz

A multi index is a vector $\alpha = (\alpha - 1, ..., \alpha_d)$ where each α_i is a non-negative integer. The notation $\alpha \geq \beta$ means $\alpha_i \geq \beta_i$ for every i. For any multi-index α, we denote

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d,\ \alpha! = \alpha_1!\alpha_2!....\alpha_d!.$$

For any vector $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$.
we denote $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_d^{\alpha_d}$ and

$$D^\alpha f(x) = D_1^{\alpha_1} D_2^{\alpha_2} ... D_d^{\alpha_d} f = \frac{\partial^{\alpha_1 + \alpha_2 + ... + \alpha_n}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} ... \partial {x_d}^{\alpha_d}} f$$

Example 2.4. With $n = 3$ and $x = (x, y, z)$, we have

$$D^{(0,2,0)} f = \frac{\partial^3 f}{\partial y^3},\ D^{(1,0,1)} f = \frac{\partial^2 f}{\partial x \partial z}, x^{(2,1,5)} = x^2 y z^5.$$

Theorem 2.2.2 *(***Multinomial Theorem:** *For any $x = (x_1, x_2, ...x_n) \in \mathbb{R}^n$ and any positive integer k,*

$$(x_1 + x_2 + x ... + x_n)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha.$$

Proof. The case of $n = 2$ is just the binomial theorem:

$$(x_1 + x_2)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x_1^j x_2^{k-j} = \sum_{\alpha_1 + \alpha_2 = k} \frac{k!}{\alpha_1!\alpha_2!} x_1^{\alpha_1} x_2^{\alpha_2} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha,$$

where we set $\alpha_1 = j, \alpha_2 = k - j$. The general case now follows by induction. Suppose the result is true for $n - 1$, i,e,. for $x = (x_1, ...x_{n-1})$. Then for $x = (x_1, ...x_n)$, we obtain

$$(x_1 + ... + x_n)^k = ((x_1 + x_2 ... + x_{N-1}) + x_N)^k$$
$$\sum_{i+j=k} \frac{k!}{i!j!} (x_1 + ... + x_{n-1})^i x_n^j$$
$$\sum_{i+j=k} \frac{k!}{i!j!} \sum_{|\beta| = i} \frac{i!}{\beta!} (x')^\beta x_n^j,$$

where $\beta = (\beta_1, ..., \beta_{N-1})$ and $x' = (x_1, ..., x_{n-1})$. To conclude, set $\alpha = (\beta_1, ..., \beta_{n-1}, j)$. Then $\beta! j! = \alpha!$ and $x' x_n^j = x^\alpha$. By noting that α runs over all multi-indices of order k when β runs over all multi-indices of order $i = k - j$ and j runs from 0 to k, we obtain $\sum_{|\alpha = k} k! \frac{x^\alpha}{\alpha!}$.
Using similar argument, one can obtain the product rule for higher order partial derivatives:

$$D^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(D^\beta f)(D^\gamma g).$$

2.2.2 System of ODE

Here we prove the Cauchy Kovalevski theorem for a system of ODE's

Definition 2.2.1 *Let f, F be realvalued functions with domain in $\mathbb{R}^n$ of class C^∞ in a neighbourhood of the origin. We say f is majorised (in symbols $f << F$), if*

$$|D^\alpha f(0)| \leq D^\alpha F(0) \text{ for all multi index } \alpha.$$

Theorem 2.2.3 *Let $f : (-a, a)^d \to \mathbb{R}^d$ is real analytic and $u(t)$ is the unique solution of the ODE*

$$u'(t) = f(u(t)),\ u(0) = 0.$$

Then u is real analytic near 0.

Lemma 2.2.1 *h is real analytic near $0 \in (-a, a)^d$. Then*

$$h(z) << \frac{Cr}{r - z_1 - ... - z_d},$$

for some C, r.

Proof. By definition, there exists $\rho > 0$ such that

$$h(z) = \sum_\alpha h_\alpha z^\alpha$$

for $|z| < \rho$ where $h_\alpha = \dfrac{1}{\alpha!}\partial^\alpha h(0)$.
Let $z = r(1, 1, ..., 1)$ with $r < \rho$, we get

$$|h_\alpha| r^{|\alpha|} \leq C, \forall \alpha$$

i.e.,

$$\begin{aligned} |h_\alpha| &\leq C r^{-|\alpha|} \\ &\leq C \frac{|\alpha|!}{\alpha!} r^{-|\alpha|}, \quad \text{(Exercise)} \end{aligned}$$

and

$$\sum_{\alpha} C \frac{|\alpha|!}{\alpha!} r^{-|\alpha|} z^{\alpha} = C \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} \left(\frac{z}{r}\right)^{\alpha}$$
$$= C \sum_{n=0}^{\infty} \left(\frac{z_1 + z_2 + ... + z_d}{r}\right)^n$$
$$= \frac{Cr}{r - z_1 - ... - z_d}$$

This completes the proof of the Lemma (2.2.1). □

Proof of the Theorem (2.2.3) Following the arguments as in scalar case,

$$f_j(z) << g_j(z) := \frac{Cr}{r - z_1 - ... - z_d}, \ \forall j$$

Let $v(t)$ be the solution of ODE

$$v'(t) = g(v(t)) = \frac{Cr}{r - v_1 - ... - v_d}(1,1,...,1), \ v(0) = 0.$$

By symmetry $v_1(t) = v_2(t) = ... = v_d(t)$. Therefore,

$$v_t'(t) = \frac{Cr}{r - dv_1}, \ v(0) = 0.$$

Therefore,

$$v_1(t) = \frac{r}{d}\left(1 - \sqrt{1 - 2Cd\frac{t}{r}}\right).$$

Therefore, $v(t) = v_1(t)(1,1,...,1)$ is convergent for $|t| < \frac{r}{2Cd}$. Now suppose, u is real analytic solution of the given problem, then we see that

$$u_j'' = \sum_{i=1}^{n} \partial_i f_j(u) u_i' = \sum_{i=1}^{n} \partial_i f_j(u) f_i(u).$$

Similarly, one can see that

$$u_j^{(n)}(t) = p_n\left(\{D^{\alpha} f_j(u)\}_{|\alpha|<n}, \{u_i^{(k)}(t)\}_{k<n, 1\leq i\leq d}\right)$$

and

$$|u_j^{(n)}(0)| \leq p_n\left(\{D^{\alpha} f_j(u)\}_{|\alpha|\leq n}\right)$$

2.3 Cauchy Kovalevski Theorem for PDE

Example 2.5. The Cauchy problem

$$w_x = \frac{Cr}{r - y - aw}(bwy + 1), \ w(0,y) = 0$$

has unique real analytic solution.

Solution: Let $g(y,z) = \frac{Cr}{r-y-az}$. Then

$$\frac{dx}{dt} = 1,\ x(0,s) = 0$$
$$\frac{dy}{dt} = -bg(y,z),\ (0,s) = 0$$
$$\frac{dz}{dt} = g(y,z),\ z(0,s) = 0.$$

Now noting that $\frac{dy}{dt} + b\frac{dz}{dt} = 0$. Integrating first two equations we get $y + bz = s, x(t,s) = t$. Therefore integrating third equation,

$$\frac{dz}{dt} = g(s-bz,z) = \frac{Cr}{r-s+bz-az} = \frac{Cr}{r-s+(b-a)z}, z(0,s) = 0.$$

Integrating this equation,

$$\begin{aligned} Crt &= \int_0^t [r-s+(b-a)z(\tau)]z'(\tau)d\tau \\ &= (r-s)z + \frac{1}{2}(b-a)z^2 \\ &= (r-y+bz)z + \frac{1}{2}(b-a)z^2 \\ &= (r-y)z - \frac{1}{2}(b-a)z^2 \end{aligned}$$

Therefore,

$$z(t) = \frac{1}{(a+b)}[(r-y) - \sqrt{(r-y)^2 - 2(a+b)Crt}]$$
$$z(x,y) = \frac{1}{(a+b)}[(r-y) - \sqrt{(r-y)^2 - 2(a+b)Crx}]$$

Example 2.6. The problem

$$v(x,y) = (v^1, v^2, ..., v^m)(x, y_1, ..., y_n)$$
$$v_x^j = \frac{Cr}{r - y_1 - ... - y_n - \sum_{k=1}^{m} v^k}[1 + \sum_{i=1}^{n}\sum_{k=1}^{m} \partial_i v^k],\ v(0,y) = 0$$

has real anlytic solution.

Solution: By symmetry, $v^j = v^1 = w(x,z)$, $z = y_1 + y_2 + ... + y_n$.

$$w_x = \frac{Cr}{r-z-mw}[mnw_z + 1],\ w(0,z) = 0.$$

By above problem, $a = m, b = mn$,

$$w(x,z) = \frac{1}{m(n+1)}[(r-z) - \sqrt{(r-z)^2 - 2m(n+1)Crx}].$$

Hence

$$v(x,y) = w(x, y_1, y_2, ..., y_n)(1,1,...,1) \in \mathbb{R}^m$$

2.3.1 First order PDE(Scalar case)

Consider the Cauchy problem

$$\frac{\partial u}{\partial x_n} = \sum_{k=1}^{n-1} \frac{\partial u}{\partial x_k} A_k(u) + B(u)$$
$$u(x',0) = 0,\ x' = (x_1, ..., x_{n-1})$$

$A_k(u)$ is real analytic at $u = 0$.

$$A_k(u) = \sum_{\alpha} D^{\alpha} A_k(0) \frac{u^{\alpha}}{\alpha!}$$

By analogous estimates in ODE, we can majorize A_k by $g(u) := \dfrac{C\rho}{\rho - u}$, where C is such that

$$\frac{1}{\alpha!} |D^{\alpha} A_k(0)| \rho^{\alpha} \leq C.$$

Then we can try to find solution of

$$\frac{\partial v}{\partial x_n} = \frac{C\rho}{\rho - v} \sum_{k=1}^{n-1} \frac{\partial v}{\partial x_k} + \frac{C\rho}{\rho - v},\ u(x',0) = 0.$$

This can be solved by method of characteristic to obtain real analytic solution. We can also take the coefficients A_k that depend on x'. That is

$$\frac{\partial u}{\partial x_n} = \sum_{k=1}^{n-1} \frac{\partial u}{\partial x_k} A_k(x_1, x_2, ..., x_{n-1}, u) + B(x_1, x_2, ..., x_{n-1}, u)$$
$$u(x_1, x_2, ..., x_{n-1}, 0) = 0.$$

In this case the majorant problem would be

$$\frac{\partial u}{\partial x_n} = \frac{C\rho}{\rho - x_1 - x_2 - ... - x_{n-1} - v} \sum_{k=1}^{n-1} \frac{\partial u}{\partial x_k} + \frac{C\rho}{\rho - x_1 - x_2 - ... - x_{n-1} - v}$$
$$u(x_1, x_2, ..., x_{n-1}, 0) = 0$$

2.3.2 Highert order PDE

Consider the problem:

$$\sum_{|\alpha| = k} a_{\alpha}(D^{k-1}u, ..., u, x) D^{\alpha} u + a_0(D^{k-1}u, ..., u, x) = 0,\ |x| < r$$
$$u = \frac{\partial u}{\partial x_n} = ... = \frac{\partial^{k-1}}{\partial x^{k-1}} = 0,\ |x'| < r, x_n = 0$$

Find $r > 0$ such that the above problem admits real analytic solution $u(x)$.
Using the transformation

$$u = u_1, \frac{\partial u}{\partial x_1} = u_2, \frac{\partial u}{\partial x_2} = u_3, ..., \frac{\partial u}{\partial x_n} = u_{n+1}$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = u_{n+2}$$

Call $U = (u_1, u_2, ..., u_{n+1}, u_{n+2}, ...)^T$, the above problem can be transformed into a system of the form

$$U_{x_n} = \sum_{j=1}^{n-1} B_j(U, x')U_{x_j} + c(U, x') \text{ for } |x| < r$$
$$U = 0 \text{ for } |x'| < r \text{ and } x_n = 0$$

where $B_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to M_{m\times m}$, $(j = 1, 2, ..., n-1)$, $B_j = (b_j^{kl})_{k,l}$ and $c : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m$, $c = (c^1, c^2, ..., c^m)$.
Here note that B is independent of x_n. x_n may be assigned u_{m+1} if B depends on it. The component equation reads,

$$u^k_{x_n} = \sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(U, x')u^l_{x_j} + c^k(U, x'),\ k = 1, 2, ..., m.$$

Theorem 2.3.1 ***Cauchy Kovalevski Theorem:*** *Assume that $\{B_j\}_{j=1}^{n-1}$ and c are real analytic functions. Then there exist $r > 0$ and a real analytic function $U = \sum_\alpha U_\alpha x^\alpha$, a solution of the above problem.*

Proof. As we noticed earlier, we must compute the coefficients $U_\alpha = \frac{D^\alpha U(0)}{\alpha!}$, in terms of $\{B_j\}_{j=1}^{n-1}$ and c. Since b_j^{kl} and c^k are analytic, we have

$$b_j^{kl} << \frac{Mr}{r - x_1 - x_2 - ... - x_{n-1} - u_1 - ... - u_m} := b_j^{kl}$$
$$c^k << \frac{Mr}{r - x_1 - x_2 - ... - x_{n-1} - u_1 - ... - u_m} := \overline{c^k}$$

Then the equation of majorant is

$$u^{*k}_{x_n} = \left(1 + \sum_{j=1}^{n-1} \sum_{l=1}^{m} u^{*l}_{x_j}\right) \frac{Cr}{r - (x_1 + x_2 + ... + x_{n-1}) - (u_1^* + ... + u_m^*)} \text{ for } |x| < r$$
$$u^* = 0 \text{ for } |x'| < r \text{ and } x_n = 0$$

By symmetry $u^{*1} = u^{*2} = ... = u^{*m}$. Therefore $u^* = v(1, 1, ..., 1)$. So by taking $s = x - 1 + x_2 + ... + x_{n-1}$, we see that v satisfies

$$v_{x_n} = \frac{Mr}{r - s - mv}(1 + m(n-1)v_s, v(s, 0) = 0.$$

This can be solved by the method of characteristics to find

$$v = \frac{1}{mn}(r - s) - [(r - s)^2 - 2mnCrx_n]^{1/2}$$

This is analytic for $|x| < r$, r small.
Claim: $0 < |u^k_\alpha| \le u^{*k}_\alpha$ for all α.
By earlier observations, we can find that

$$u^k_\alpha = q^k_\alpha(B_{j,\nu,\delta}, c_{\nu,\delta}, u_\beta)$$

q^k_α is a polynomial.

$$\begin{aligned}|u^k_\alpha| &= |q^k_\alpha(...,B_{j,\nu,\delta}(u),c_{\nu,\delta}(u),u_\beta)| \\ &\leq q^k_\alpha(...,B_{j,\nu,\delta}(u^*),c_{\nu,\delta}(u^*),u^*_\beta) \\ &= u^{*k}_\alpha\end{aligned}$$

Hence the claim and the proof of the theorem. □

As we mentioned earlier, this theorem only guarantees the existence and uniqueness, but not well-posedness. The following problem explains this:

Example 2.7. Consider the Cauchy problem:

$$u_{xx}+u_{yy}=0,\ \ u(x,0)=0,u_y(x,0)=\frac{sinkx}{k}$$

It is easy to see the solution is

$$u(x,y)=\frac{1}{k^2}\sin kx\sinh(ky)$$

Now notice that the Cauchy data converges uniformly to 0 as $k\to 0$, but the solution does not converge to 0 at any $y\neq 0$.

Example 2.8. Consider the parabolic equation

$$u_t=u_{xx},\ \ u(x,0)=\frac{1}{1-x^2},\ t>0,x\in\mathbb{R}.$$

It is easy to see that $t=0$ is a characteristic line as $A=1,B=0,C=0$. If we write the solution in the form

$$u(t,x)=\sum a_{m,n}\frac{t^m}{m!}\frac{x^n}{n!}$$

Then substituting in the PDE, we get

$$a_{m+1,n}=a_{m,n+2}$$

$$u(x,0)=\sum a_{0,n}\frac{x^n}{n!}=1+x^2+x^4+....$$

$$a_{0,2n=1}=0,\ \ a_{0,2n}=(2n)!$$

hence we get

$$a_{m,2n}=a_{0,2n+2m}=[2(n+m)]!$$

Therefore,

$$\frac{a_{m,2n}}{m!(2n)!}=\frac{[2(n+m)]!}{m!(2n)!}$$

Taking $n=m$ we see

$$\frac{a_{m,2n}}{m!(2n)!}\sim\frac{(4n)^{4n}}{n^n(2n)^{2n}}$$

It is not difficult to show that the radius of convergence is zero.

2.3.3 Holmgren Uniqueness theorem

In the C-K theorem, we proved the existence of unique real analytic solution for Cauchy problem in with real analytic data. Then one asks the question wheather the problem has another solution which is differentiable of the order of the equation. This is answered by the Holmgren uniqueness theorem. This theorem says that for Linear problem with real analytic data, there does not exist another C^m solution. The surprising thing is it is the C-K theorem which is used to prove the Holmgrent uniqueness theorem. Consider the Cauchy problem for linear differential operator with data on the initial curve γ

$$(P): \ \mathscr{L}u = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = 0, \quad D^\alpha u = 0 \text{ on } \gamma \text{ for } |\alpha| \le m-1.$$

The main idea of the proof is to show that the adjoint problem has real analytic solution. This existence implies the uniqueness of solution to the given problem. This is popularly known as "Existence implies uniqueness". To move forward, let us define the adjoint $\mathscr{L}^t$ of the operator $\mathscr{L}$ as

$$\mathscr{L}^t v = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \big(a_\alpha(x) v\big).$$

Then one can easily see by integration by parts,

$$\langle \mathscr{L}u, v\rangle = \int_\Omega \mathscr{L}u\, v = \int_\Omega u\, \mathscr{L}^t v + \int_{\partial\Omega} a_\alpha(x) D^\alpha u\, D^\alpha v d\sigma = \langle u, \mathscr{L}^t v\rangle + \text{Boundary terms},$$

The boundary terms will be zero if $D^\alpha u = 0$ or $D^\alpha v = 0$ on $\partial\Omega$. So if we choose $u, v \in C^m(\Omega)$ such that either $D^\alpha u = 0$ or $D^\alpha v = 0$ on $\partial\Omega$, then we have

$$\langle \mathscr{L}u, v\rangle = \langle u, \mathscr{L}^t v\rangle$$

The main motivation comes from the adjoint operators on normed linear spaces. Let X, Y be normed linear spaces and let X', Y' are the normed duals. Let $\mathscr{L}: X \to Y$ be a continuous map.

Definition 2.3.1 *The adjoint of $\mathscr{L}$ is defined as $\mathscr{L}^t: Y' \to X'$ by*

$$\langle \mathscr{L}^t y', x\rangle = \langle y', \mathscr{L}x\rangle \ \textit{for all } y' \in Y' \textit{ and } x \in X.$$

Theorem 2.3.2 *If the range of $\mathscr{L}^t$ is dense in X', then* $\ker(\mathscr{L}) = \{0\}$.

Proof. Let $\mathscr{L}u = 0$. Then

$$0 = \langle y', \mathscr{L}u\rangle = \langle \mathscr{L}^t y', u\rangle$$

i.e. $\langle \mathscr{L}^t y', u\rangle = 0$ for all $y' \in Y'$. Since $\mathscr{L}^t$ has dense range it implies that

$$\langle x', u\rangle = 0 \ \ \forall\, x' \in X'.$$

By Hahn Banach theorem it follows that $u = 0$.

Now construct γ' so that γ and γ' are non-characteristic curves for the problem (P). Let Ω be the domain bounded by curves γ and γ' and the boundary $\partial\Omega = \gamma \cup \gamma'$. We may consider the operator $X = C^m(\overline{\Omega})$ and $Y = C(\overline{\Omega})$ with metrics by

$$d_m(u,v) = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v dx, \quad d(u,v) = \int_\Omega uvdx.$$

and the operator $\mathscr{L} : C^m(\overline{\Omega}) \to C^(\overline{\Omega})$. Then for $u,v \in C^m(\overline{\Omega})$ satisfying

$$D^\alpha u = 0 \text{ on } \gamma \text{ and } D^\alpha v = 0 \text{ on } \gamma', \text{ for all } |\alpha| \le m-1,$$

we have $\langle \mathscr{L} u \cdot v\rangle = \langle \mathscr{L}^t v \cdot u\rangle$. Therefore, by above Theorem, if the range of $\mathscr{L}^t$ is dense in $C(\overline{\Omega})$, then $\mathscr{L} u = 0$ implies $u = 0$. That is, we need the existence of solution for following adjoint problem

$$(Q_g): \ \mathscr{L}^t v = g, \quad D^\alpha v = 0 \text{ on } \gamma', \text{ for } |\alpha| \le m-1$$

for all g in a dense subset of $C(\overline{\Omega})$. By Weierstrass approximation theorem, we can choose a sequence $\{g_n\}$ of polynomials that converges to g in $C(\overline{\Omega})$. So it is enough to show the problem in (Q_g) has solution for all polynomials g and the domain of existence of solution does not depend on g. Since polynomils are real analytic in the whole space and since $\mathscr{L}$ is linear operator, by Cauchy-Kowalevski theorem, this problem has unique real analytic solution in a neighborhood around γ'. The size of the neighbourhood does not depend on g (exercise). But it depends on the coefficients of $\mathscr{L}$.
In more simple terms, if u is a solution of (P) and let v_n is a solution of (Q_{p_n}). Then

$$0 = \langle \mathscr{L} u, v_n\rangle = \langle u, \mathscr{L}^t v_n\rangle = \langle u, g_n\rangle.$$

If we choose g_n such that g_n is polynomial and $g_n \to u$ uniformly in $C(\overline{\Omega})$, we have

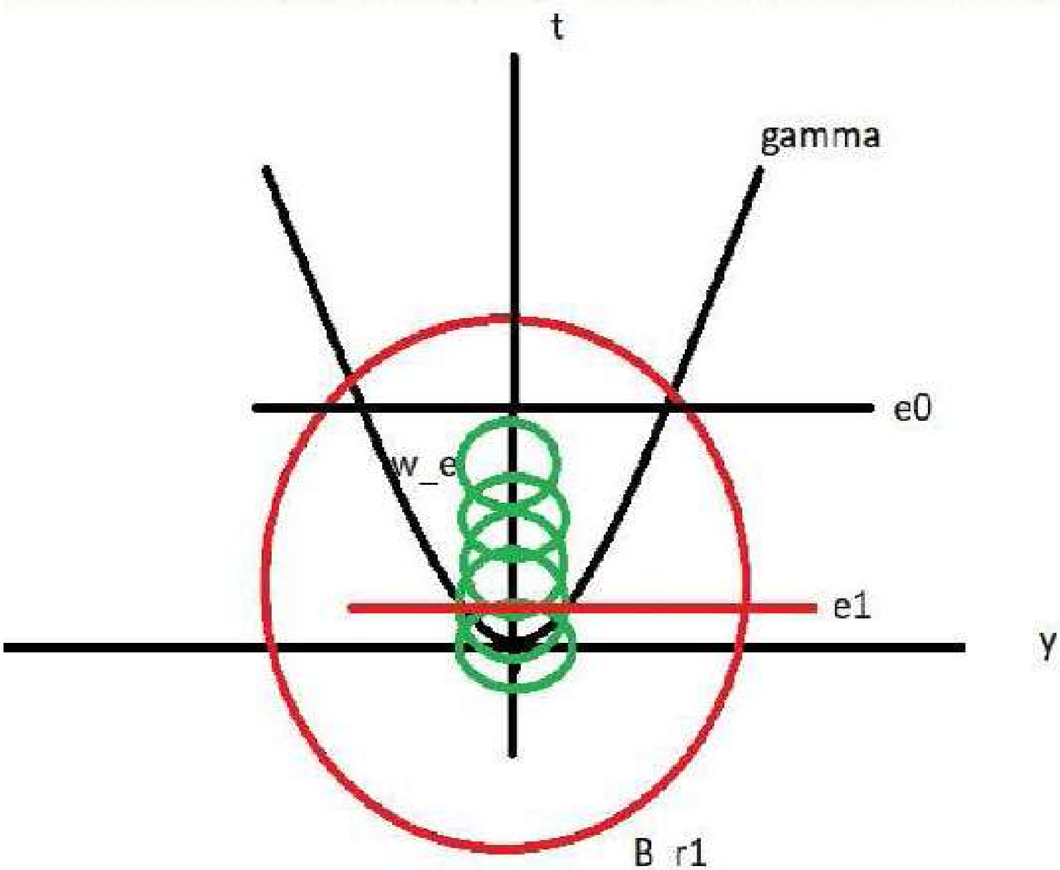
$$0 = \int u\, g_n \to \int u^2 \implies u = 0.$$

Now we demostrate these constructions for Cauchy problem: Consider the Cauchy problem on $\gamma = \{x = (x_1,\dots,x_d) \in \mathbb{R}^d : x_1 = 0\}$, the non-characteristic initial curve and let $x_0 = (0,\dots,0)$. Introduce the new coordinates $(t, y_2,\dots,y_d)$, where

$$t = x_1 + x_2^2 + \cdots + x_d^2,\ y_2 = x_2,\ y_3 = x_3,\dots,y_d = x_d.$$

Then, $\gamma = \{t = |y|^2\}$. With respect to this new coordinates, γ is non-characteristic for the Cauchy problem of the operator $\mathscr{L} u = \sum a_\alpha(t,y) D^\alpha u$, and assume that there exists r_1 such that the coefficients $a_\alpha(t,y)$ are real analytic in $B_{r_1} = \{(t,y) : |t| + |y| < r_1\}$. So, choose ε_0 such that

$$w_{\varepsilon_0} = \{|y|^2 < t < \varepsilon_0\} \Subset B_{r_1}.$$



There exists constant C such that

$$\frac{|D^{\alpha} a_{\alpha}(t,y)|}{\alpha!} \leq C\rho^{\alpha}, \ \ |t|+|y| < \rho < r_1.$$

The coefficients b_{α} of the adjoint operator are also analytic and so we may assume that

$$\frac{|D^{\alpha} b_{\alpha}(t,y)|}{\alpha!} \leq C\rho^{\alpha}, \ \ |t|+|y| < \rho < r_1.$$

For each $\varepsilon \in [0, \varepsilon_0]$, we consider the Cauchy problem of adjoint

$$\mathscr{L}^t v = g_n(t,y),\ t \in (0,\varepsilon), \quad D^{\alpha} v = 0 \quad \text{on } \{t = \varepsilon\}, |\alpha| \leq m-1,$$

where g_n is a polynomial. By Cauchy-Kowalevski theorem, there exists a neighborhood around $(\varepsilon, 0)$, $\varepsilon < \varepsilon_0$ with uniform radius of convergence ρ as g_n is polynomial. Therefore, there exists solution in $\{(t,y) : |t-\varepsilon|+|y| < \rho\}$ for each $\varepsilon \in [0, \varepsilon_0]$.

Now it is easy to see that there exists ε_1 such that $w_{\varepsilon_1} \subset \{|t-\varepsilon_1|+|y| < \rho\}$. Therefore, $\mathscr{L}^t$ has dense range on $C^m(\overline{w}_{\varepsilon_1})$. By adopting this at each non-characteristic point x_0 on γ we obtain that $u = 0$ on one side of γ. The argument for the other side is identical.

Reference:

1. Lectures on Cauchy problem by Sigeru Mizohate, TIFR Bombay, 1965.

2. Lecture notes of Gustav Holzegel, Imperial college, London.

2.3.4 Problems

1. $\sum_{\alpha} x^{\alpha} = \Pi_{i=1}^{n} \left(\sum_{\alpha_i=0}^{\infty} x_i^{\alpha_i} \right) = \frac{1}{(1-x_1)(1-x_2)....(1-x_n)}$
2. $\sum_{\alpha} \frac{|\alpha|!}{\alpha!} x^{\alpha} = \sum_{j=0}^{\infty} \sum_{|\alpha=j} \frac{|\alpha|!}{\alpha!} x^{\alpha} = \sum_{j=0}^{\infty} (x_1 + ... + x_n)^j = \frac{1}{1-(x_1+x_2...+x_n)}$ provided $|x_1 + ...|x_n| < 1$.
3. For $x, y \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, \quad (x+y)^{\alpha} = \sum_{\beta,\gamma\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^{\beta} y^{\gamma}$
4. Taylor expansion: $f(x) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!} (D^{\alpha} f(0)) x^{\alpha}$.
5. Solve the Cauchy problem $u_t = \frac{Mr^2(n-1)}{(r-u)(r-t)} u_s,\ u(s,0) = s$.
6. In the linear case, the Cachy problem

$$\frac{\partial u}{\partial y} = A(x,y)\frac{\partial u}{\partial x} + B(x,y)u + c(x)$$

can be majorized by

$$\frac{\partial u}{\partial y} = \frac{r^2}{(r-x)(r-y)} \left(M\frac{\partial u}{\partial x} + Mu + \mu \right)$$

for suitable constants r, M, μ. Solve this problem with Cauchy data $u(x,0) = 0$. Show that the region in which solution exists is independent of μ.
7. Show that $u(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t}$ is a smooth solution of equation $u_t = u_{xx}$ for $t > 0, x \in \mathbb{R}$. Extend u by zero for $t < 0$. Does this contradict the Holmgren uniqueness theorem?.
8. Consider the Cauchy problem $u_{tt} = u_{xx}, t > 0, x \in \mathbb{R},\ u(x,0) = 0$. Does Holmgren uniqueness theorem imply that $u \equiv 0$.