Chapter 6
Wave Equation

In this section we will study the wave equation

$$u_{tt} - c^2 \Delta u = 0,\ t > 0,\ x \in \mathbb{R}^n.$$

First we will try to find a solution of the Cauchy problem in one dimension. This models the position of vibrating string. Firstly, we try to solve the problem with given Cauchy data $f(x), g(x)$:

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R},\ t > 0$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x),\ x \in \mathbb{R}.$$

The characteristics of the equation are

$$x + ct = c_1,\ x - ct = c_2$$

where c_1, c_2 are constants. Now taking the transformation

$$\eta = x - ct,\ \xi = x + ct$$

we get the following

$$u_\eta = \frac{-1}{2c} u_t + \frac{1}{2} u_x,\ \ u_\xi = \frac{1}{2c} u_t + \frac{1}{2} u_x$$
$$u_{\xi\eta} = \frac{1}{4}\left(-\frac{1}{c^2} u_{tt} + u_{xx} \right) = 0.$$

Therefore,

$$u(\eta, \xi) = F(\xi) + G(\eta) = F(x + ct) + G(x - ct)$$

for some functions F and G. To find these functions we use the initial conditions

$$f(x) = u(x,0) = F(x) + G(x),\ g(x) = u_t(x,0) = c(F'(x) - G'(x))$$

solving these two equations, we get

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds)$$

This is known as **D'Alembert's formula.** In case of non-homogeneous equation

$$u_{tt} - c62u_{xx} = h(x,t)$$

with the transformation $(x,t) \mapsto (\eta, \xi)$, we get

$$u_{\eta\xi} = -\frac{1}{4c^2}(u_{tt} - c^2 u_{xx}) = -\frac{1}{4c^2} h(\eta, \xi)$$

Therefore,

$$u(\xi, \eta) = \int \int h(\eta, \xi) d\xi d\eta + F(\xi) + G(\eta)$$

where F and G are evaluated using the Cauchy data.

Example 6.1. solve the problem

$$u_{tt} - c^2 u_{xx} = \cos t, u(x,0) = 0, u_t(x,0) = 0, \; x \in \mathbb{R}, \; t > 0.$$

In the variables η and ξ

$$u_{\eta\xi} = -\frac{1}{4c^2} h(\eta, \xi) = -\frac{1}{4c^2} \cos\left(\frac{\xi - \eta}{2c}\right), \; t = \frac{\xi - \eta}{2c}$$

Therefore,

$$u(\xi, \eta) = \int \int h(\eta, \xi) d\xi d\eta + F(\xi) + G(\eta) = -\cos\left(\frac{\xi - \eta}{2c}\right) + F(\xi) + G(\eta) = -\cos t + F(\xi) + G(\eta)$$

Now substituting the initial conditions $u(x,0) = 0, u_t(x,0) = 0$, we get

$$F + G = 0, \; F' - g' = 0$$

This implies $F = \frac{1}{2} + k, G = \frac{1}{2} - k$ for some constant k. Substituting in $u(x,t)$ we get

$$u(x,t) = 1 - \cos t.$$

In case of Initial Boundary value problems on $(0,l)$:

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \; t > 0, x \in (0,l) \\ u(x,0) = f(x), \, u_t(x,0) = g(x), \; x \in (0,l) \\ u(0,t) = 0, \, u(l,t) = 0. \end{aligned}$$

Then in prinicple the solution should satisfy the D'Alemberts formula. But when t, time is large enough, $x + ct$ and $x - ct$ are not in the interval $(0,l)$. So for this formula to be valid, we need to extend the functions $f(x)$ and $g(x)$ to the whole of $\mathbb{R}$.
Let $\overline{f}(x)$ and $\overline{g}(x)$ be the extensions of $f(x)$ and $g(x)$ to the whole of $\mathbb{R}$. Similarly, $\overline{F}$ and $\overline{G}$ be extension of F and G to $\mathbb{R}$. Then from the boundary conditions, we get

$$\begin{aligned} 0 &= u(0,t) = \overline{F}(ct) + \overline{G}(-ct) \\ &= \frac{1}{2}\left[\overline{f}(ct) + \overline{f}(-ct)\right] + \frac{1}{2c}\int_0^{ct} \overline{g}(s) ds - \frac{1}{2c}\int_0^{-ct} \overline{g}(s) ds \end{aligned}$$

Thereofe $\overline{f}(ct) = -\overline{f}(-ct)$ and $\int_{-ct}^{ct} \overline{g}(s)ds = 0$. This implies that the extended function is an odd function. Also, from the other boundary condition we get,

$$0 = u(l,t) = \frac{1}{2}\left[\overline{f}(l+ct) + \overline{f}(l-ct)\right] + \frac{1}{2c}\int_{l-ct}^{l+ct} \overline{g}(s)ds$$

This implies that

$$\overline{f}(l+ct) = -\overline{f}(l-ct) = \overline{f}(-l+ct)$$

where we used that $\overline{f}$ is odd for last equality. Similarly for $\overline{g}$. Therefore $\overline{f}$ and $\overline{g}$ are **odd periodic extensions of period** $2l$.

Parallologram property: The solution of the wave equation satisfies parallelogram property. Let $u(\eta,\xi) = F(\xi) + G(\eta)$ is the general solution. Suppose ABCD is a rectangle in the $\eta\xi-$plane having sides parallel to the coordinate axes with coordinates:

$$A = (\xi + \delta\xi, \eta),\ B = (\xi, \eta),\ C = (\xi, \eta + \delta\eta),\ D = (\xi + \delta\xi, \eta + \delta\eta)$$

Then F is constant along CD and AB. G is constant along BC and AD. Then

$$\begin{aligned}
u(A) &= F(\xi + \delta\xi) + G(\eta)\\
u(B) &= F(\xi) + G(\eta)\\
u(C) &= F(\xi) + G(\eta + \delta\eta)\\
u(D) &= F(\xi + \delta\xi, \eta) + G(\eta + \delta\eta)
\end{aligned}$$

Therefore,

$$u(B) + u(D) = u(A) + u(C)$$

This rectanle is transformed into a parallelogram in the xt-plane with sides paralle to the characteristic lines $x = ct$ and $x = -ct$. This is known as the parallelogram property satisfied by the wave equation. This motivates one to define the weak solution.

Definition 6.0.1 *Any function $u(x,t)$ (need not be even continuous) satisfying the parallelogram property is called weak solution.*

Example 6.2. Solve the following initial boundary value problem using D'Alemberts formula:

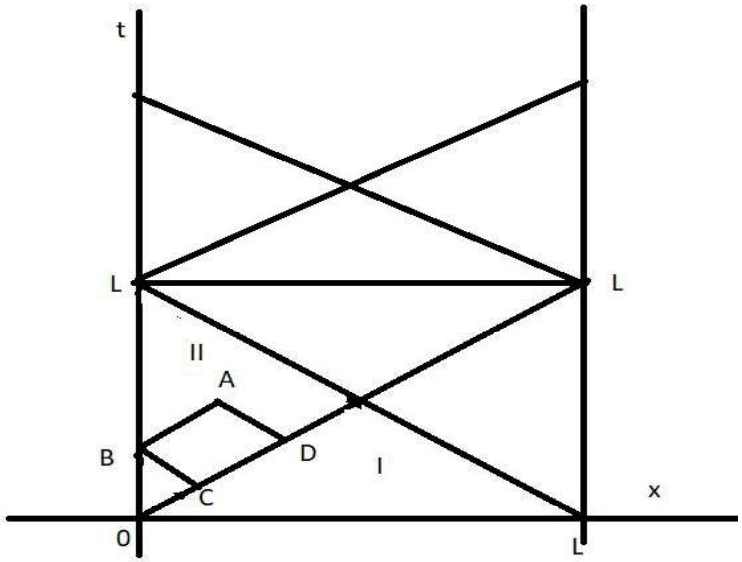
$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0,\ t > 0, x \in (0,L)\\
u(x,0) &= x,\ u_t(x,0) = 0,\ x \in (0,L)\\
u(0,t) &= \alpha(t),\ u(L,t) = 0,\ t \geq 0.
\end{aligned}$$

In the region (I),

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) = \frac{1}{2}(x + ct + x - ct) = x.$$

In the region (II), we use the parallelogram property,

$$u(A) = -u(C) + u(B) + u(D)$$



Let $A = (x_0, t_0)$. Then the line AD: $x - x_0 = (-1)(t - t_0)$ and this line intersects $x = t$ at $t = \frac{x_0 + t_0}{2}$. Therefore the point $D = (\frac{x_0+t_0}{2}, \frac{x_0+t_0}{2})$.
The line AB: $x - x_0 = t - t_0$. This line intersects $x = 0$ at the point B. Therefore $B = (0, t_0 - x_0)$.
The line BC: $x = (-1)(t - t_0 + x_0)$ and this intersects the line $x = t$ at C. Therefore the point $C = (\frac{t_0 - x_0}{2}, \frac{t_0 - x_0}{2})$. The point $B = (0, t_0 - x_0)$. Therefore,

$$u(B) = \alpha(t_0 - x_0),\ u(D) = \frac{1}{2}(x_0 + t_0),\ u(C) = \frac{1}{2}(t_0 - x_0).$$

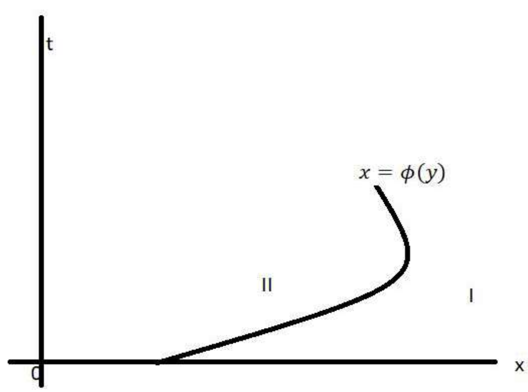
Hence,

$$u(A) = u(x_0, t_0) = x_0 + \alpha(t_0 - x_0).$$

Propagation of jump discontinuities: Consider the second order equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0.$$

Suppose u is a solution and the second order partial derivatives have discontinuous along $x = \phi(y)$ and



the jump in the first order derivatives is zero. Let u^I be the solution in the region that is left of the curve $x = \phi(y)$ and u^{II} be the solution in the region that is right of the curve $x = \phi(y)$. Then

$$\begin{aligned}0 = \frac{d}{dy}[u_x] &= \frac{d}{dy}\left(u_x^I(\phi(y),y) - u_x^I(\phi(y),y)\right)\\ &= u_{xx}^{II}\phi'(y) + u_{xy}^{II} - u_{xx}^I\phi' - u_{xy}^{II}\\ &= \phi'(u_{xx}^{II} - u_{xx}^I) + (u_{xy}^{II} - u_{xy}^I)\\ &= \phi'[u_{xx}] + [u_{xy}].\end{aligned}$$

Similarly,

$$\frac{d}{dy}[u_y] = 0 = \phi'[u_{yx}] + [u_{yy}]$$

From the given differential equation we get

$$a[u_{xx}] + 2b[u_{xy}] + c[u_{yy}] = 0.$$

Now assuming jump in u_{xy} and u_{yx} are same and let $\lambda = [u_{xx}]$. Then from the equations we get

$$[u_{xy}] = -\lambda\phi',\ [u_{xy}] = \lambda(\phi')^2.$$

Therefore,

$$a\lambda - 2b\lambda\phi' + c\lambda(\phi')^2 = 0.$$

If $\lambda \neq 0$, then ϕ satisfies

$$a - 2b\phi' + c(\phi')^2 = 0.$$

which implies the $\phi(y)$ is a characteristic curve. Therefore we say the **discontinuities propagate along the characteristic curves.** Now differentiating the equation $\lambda = [u_{xx}]$ and $\lambda\phi' + [u_{xy}] = 0$ with respect to y, we get

$$\begin{aligned}\frac{d\lambda}{dy} &= [u_{xxx}]\phi' + [u_{xxy}]\\ -(\phi'\lambda)' &= [u_{xxy}]\phi' + [u_{xyy}]\end{aligned}$$

Differentiating the given differential equation with respect to x, we get

$$a[u_{xxx}] + 2b[u_{xxy}] + c[u_{xyy}] = 0.$$

Eliminating $[u_{xxx}], [u_{xxy}]$ and $[u_{xyy}]$ from the above 3 equations, we get the first order equation in λ

$$2\lambda'(b - c\phi') - \lambda c\phi'' = 0$$

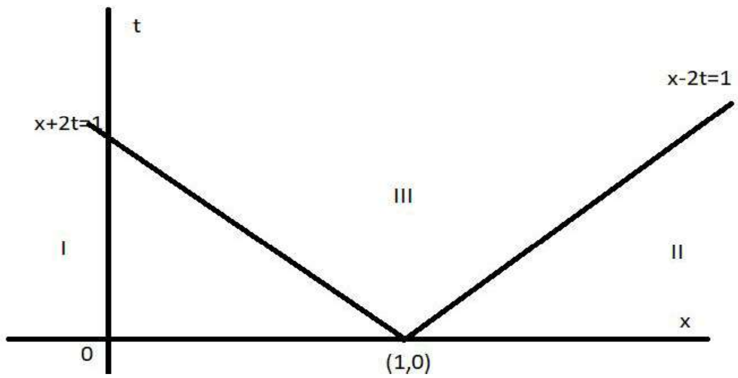
So if λ is not zero at a point then λ is never zero.

Example 6.3. Find the weak solution of the following problem: For $x \in \mathbb{R}, t > 0$, consider the Cauchy problem

$$u_{tt} - 4u_{xx} = 0,\ u(x,0) = 0, u_t(x,0) = \begin{cases} 1 & x \le 1 \\ x & x > 1. \end{cases}$$

By D'Alembert's formula,

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} g(s)ds$$



In the regioin (I):
For $(x,t) \in I$, we have $x+2t < 1, t > 0$. Therefore,

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} 1ds = t.$$

In the regioin (II):
For $(x,t) \in II$, we have $x-2t > 1, t > 0$. Therefore,

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} sds = xt.$$

In the regioin (III):
For $(x,t) \in III$, we have $x-2t < 1, x+2t > 1, t > 0$. Therefore,

$$u(x,t) = \frac{1}{4}\left(\int_{x-2t}^{1} 1ds + \int_{1}^{x+2t} sds\right) = \frac{1}{4}(x+2t-1).$$

Theorem 6.0.1 Uniqueness: *The IBVP:*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t),\ t > 0, x \in (0,L) \\ u(x,0) = f(x),\ u_t(x,0) &= g(x),\ x \in (0,L) \\ u(0,t) = a(t),\ u(L,t) &= b(t),\ t \ge 0. \end{aligned}$$

has at most one solution.

Proof. Let u_1 and u_2 be two solutions, then $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the problem

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0,\ t > 0, x \in (0,L) \\ w(x,0) &= 0,\ w_t(x,0) = 0,\ x \in (0,L) \\ w(0,t) &= 0,\ w(L,t) = 0,\ t \geq 0. \end{aligned}$$

Consider the function

$$E(t) = \frac{1}{2} \int_0^L (w_t^2 + c^2 w_x^2) dx$$

Then

$$E'(t) = \int_0^L (w_t w_{tt} + c^2 w_x w_{xt}) dx$$

Also $c^2 w_x w_{xt} = c^2 \left(\frac{\partial}{\partial x}(w_x w_t) - w_{xx} w_t \right) = c^2 \frac{\partial}{\partial x}(w_x w_t) - w_{tt} w_t$. Therefore,

$$E'(t) = \int_0^L c^2 \frac{\partial}{\partial x}(w_x w_t) dx = c^2 w_x w_t \Big|_0^L = 0.$$

Here we used $w(0,t) = 0, w_x(L,t) = 0$. This implies $E(t)$ is constant. But $E(0) = 0$. Hence $E(t) \equiv 0$. That is

$$w_t^2 + c^2 w_x^2 = 0.$$

Therefore $w_t(x,t) = 0, w_x(x,t) = 0$. Hence $w(x,t)$ is constant. Finally using $w(x,0) = 0$ we get $w(x,t) \equiv 0$.

6.0.1 Higher dimensions

Let $h(x)$ be a continuous function on $\mathbb{R}^n$, let

$$M_h(x,r) = \frac{1}{w_n} \int_{|\xi|=1} h(x + r\xi) dS_\xi$$

be the spherical mean of $h(x)$ on a sphere of radius r and center x. Here w_n is the surface area of unit sphere S^{n-1} and dS_ξ denotes the surface measure. Since $h(x)$ is a continuous function, $M_h(x,r)$ is continuous in x and r. Taking limit $r \to 0$, we get $M_h(x,0) = h(x)$. Moreover if $h \in C^k(\mathbb{R}^n)$ then $M_h(x,r) \in C^k(\mathbb{R}^n \times \mathbb{R}^n)$. Differentiaing M_h with respect to r partially, we get

$$\frac{\partial}{\partial r} M_h(x,r) = \frac{1}{w_n} \sum_{i=1}^n \int_{|\xi|=1} h_{x_i} \xi dS_\xi$$

Integrating by parts on $|\xi| < 1$ (unit exterior normal $\hat{n} = \xi$), we get

$$\begin{aligned} \frac{\partial}{\partial r} M_h(x,r) =& \frac{1}{w_n} \sum_{i=1}^n \int_{|\xi|<1} \frac{\partial}{\partial \xi_i} \left(h_{x_i}(x + r\xi) \right) d\xi \\ &= \frac{1}{w_n} \sum_{i=1}^n \int_{|\xi|<1} h_{x_i x_i}(x + r\xi) r d\xi \\ &= \frac{r}{w_n} \Delta_x \int_{|\xi|<1} h(x + r\xi) d\xi \end{aligned}$$

Now by the change of variable $\xi' = r\xi$, we get $d\xi' = r^n d\xi$ and

$$\begin{aligned}\int_{|\xi|<1} h(x+r\xi)d\xi &= \frac{1}{r^n}\int_{|\xi'|<r} h(x+\xi')d\xi' \\ &= \frac{1}{r^n}\int_0^r \rho^{n-1}\int_{|\xi|=1} h(x+\rho\xi)dS_\xi d\rho \\ &= \frac{w_n}{r^n}\int_0^r \rho^{n-1}M_h(x,\rho)d\rho\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial r}M_h(x,r) = \frac{1}{r^{n-1}}\Delta_x\int_0^r \rho^{n-1}M_h(x,\rho)d\rho$$

Differentiating with respect to r implies

$$\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}M_h(x,r)\right) = \Delta_x M_h(x,r)r^{n-1}$$

That is equivalent to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u(x,r) = \Delta_x M_h(x,r).$$

This is called Darboux equation.

Cauchy problem in n-dimensions:
Now consider the problem

$$\begin{aligned}u_{tt} &= c^2\Delta u,\ x\in\mathbb{R}^n,\ t>0 \\ u(x,0) = g(x),\ u_t(x,0) &= h(x),\ x\in\mathbb{R}^n.\end{aligned}$$

From the above discussion, we see that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}M_u(x,r,t) &= \frac{1}{w_n}\int_{|\xi|=1} u_{tt}(x+r\xi,t)dS_xi \\ &= \frac{1}{w_n}\int_{|\xi|=1} c^2\Delta u(x+r\xi,t)dS_xi \\ &= c^2\Delta_x M_u(x,r,t)\end{aligned}$$

Thereofre from the above Darboux equation, we get

$$\frac{\partial^2}{\partial t^2}M_u(x,r,t) = \left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u(x,r,t) \tag{0.1}$$

This is called Euler-Poisson-Darboux equation. The initial conditions become

$$M_u(x,r,0) = M_g(x,r),\ \ \frac{\partial}{\partial t}M_u(x,r,t) = M_h(x,r).$$

We can solve this Cauchy problem in the varibles t and r to obtain $M_u(x,r,t)$ and $u(x,t)$ may be obtained as limit

$$u(x,t) = \lim_{r\to 0}M_u(x,r,t).$$

Wave equation in $\mathbb{R}^3$
From the equation (0.1) with $n=3$, we obtain

$$\frac{\partial^2}{\partial t^2}M_u(x,r,t) = \frac{c^2}{r}\frac{\partial^2}{\partial r^2}\left(rM_u(x,r,t)\right) \tag{0.2}$$

At each x, let $V^x(r,t) = rM_u(x,r,t)$. Then V^x satisfies the IBVP:

$$\frac{\partial^2}{\partial t^2}V^x(r,t) = c^2\frac{\partial^2}{\partial r^2}V^x(r,t),\ r>0, t>0 \tag{0.3}$$

$$V^x(r,0) = rM_g(x,r),\ V^x_t(r,0) = rM_h(x,r) \tag{0.4}$$

$$V^x(0,t) = \lim_{r\to 0} rM_u(x,r,t) = 0.u(x,t) = 0. \tag{0.5}$$

Also by defining $G^x(r) = rM_g(x,r)$ and $H^x(r) = rM_h(x,r)$ we see that G and H satisfies the compatibility condition $G^x(0) = H^x(0)$.

Therefore by D'Alembert's formula, G^x and H^x may be extended as odd functions to negative values of r to write the solution as

$$V^x(r,t) = \frac{1}{2}\left(G^x(r+ct)+G^x(r-ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} H(\rho)d\rho.$$

Since G^x and H^x are odd functions of r, we have $G^x(r-ct) = -G(ct-r)$ and

$$\int_{r-ct}^{r+ct} H(\rho)d\rho = \int_{-(ct-r)}^{ct+r} H(\rho)d\rho = \int_{-(ct-r)}^{ct-r} H(\rho)d\rho + \int_{ct-r}^{ct+r} H(\rho)d\rho = 0 + \int_{ct-r}^{ct+r} H(\rho)d\rho$$

Therefore,

$$M_u(x,r,t) = \frac{1}{r}V^x(r,t) = \frac{1}{2r}\left(G^x(r+ct)+G^x(ct-r)\right) + \frac{1}{2cr}\int_{ct-r}^{r+ct} H(\rho)d\rho$$

That is,

$$M_u(x,r,t) = \frac{1}{2r}\left((r+ct)M_g(x,ct+r) - (ct-r)M_g(x,ct-r)\right) + \frac{1}{2cr}\int_{ct-r}^{r+ct} \rho M_h(x,\rho)d\rho$$

Taking limt $r\to 0$, we get

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial \tau}(\tau M_g(x,\tau))\Big|_{\tau=ct} + \frac{1}{c}\left(ctM_h(x,ct)\right) \\ &= \frac{\partial}{\partial t}(tM_g(x,ct))\Big|_{\tau=ct} + t\left(ctM_h(x,ct)\right) \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)dS_\xi \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{B(x,ct)} g(\xi)dS_\xi\right) + \frac{t}{4\pi}\int_{B(x,ct)} h(\xi)dS_\xi. \end{aligned}$$

This is known as Kichhoff's formula.

Remark 6.0.1 *1. The domain of dependence is the sphere of radius ct around x. That is, the surface of the sphere $\{x+ct\xi : |\xi| = 1\}$ in $\mathbb{R}^3$. Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the set of points on the cone $\{(x,t) : |x-x_0| = ct\}$. This phenomena is called presence of sharp signals. That is, A large initial disturbance near x_1 will be felt at time $t = \frac{|x_1|}{C}$ and not after that time. For example sound or light waves.*

2. *The loss of regularity: The formula above implies that $g \in C^3, h \in C^2$. Then $u \in C^2$. That is if the initial data is of C^{k+1}, then solution is C^k. So there is a loss of regularity. This occurs only in higher dimensions and not in one dimension.*

Wave equation in $\mathbb{R}^2$

This method is also known as **Hadamard method of descent.** In this case it is not possible to write the equation (0.2). So, in this case we regard $\mathbb{R}^2$ as $\mathbb{R}^3$ with $x_3 = 0$. Therefore we need to convert the surface integrals in $\mathbb{R}^3$ to domain integrals in $\mathbb{R}^2$. Let $u(x_1, x_2, t)$ be the required solution. Then define

$$\overline{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t).$$

Then the Cauchy problem becomes

$$\overline{u}_{tt} - \Delta \overline{u} = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \tag{0.6}$$

$$\overline{u} = \overline{g},\ \overline{u_t} = \overline{h} \text{ on } \mathbb{R}^3 \times \{t = 0\} \tag{0.7}$$

where $\overline{g}(x_1, x_2, x_3) = g(x_1, x_2)$, $\overline{h}(x_1, x_2, x_3) = h(x_1, x_2)$. Denoting $\overline{x} = (x_1, x_2, 0) \in \mathbb{R}^3$ for $x \in \mathbb{R}^2$. Then the solution of (0.6) -(0.7) is

$$u(x,t) = \overline{u}(\overline{x},t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \overline{B}(\overline{x},t)} \overline{g}\, d\overline{s}\right) + t \int_{\partial \overline{B}(\overline{x},t)} \overline{h}\, d\overline{s}$$

where $\overline{B}(\overline{x},t)$ is the ball in $\mathbb{R}^3$ with center $\overline{x}$ and radius > 0 and $d\overline{s}$ denotes the two dimensional surface measure on $\partial \overline{B}(\overline{x},t)$. To simplify further, we note that $\partial \overline{B}(\overline{x},t) = \{y : |y - \overline{x}| = t\} = \{(y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^2 = t^2\}$. Now

$$\fint_{\partial \overline{B}(\overline{x},t)} \overline{g}\, d\overline{s} = \frac{1}{4\pi t^2} \int_{\partial \overline{B}(\overline{x},t)} \overline{g}\, d\overline{s} = \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |D\gamma(y)|)^{1/2} dy$$

where $\gamma(y) = (t^2 - |y - x|^2)^{1/2}$ for $y \in B(x,t)$. The factor 2 is because the sphere has upper and lower hemispheres. also

$$(1 + |D\gamma|^2)^{1/2} = t\left(t^2 - |y - x|^2\right)^{-1/2}.$$

Therefore,

$$\begin{aligned} \fint_{\partial \overline{B}(\overline{x},t)} \overline{g}\, d\overline{s} &= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y - x|^2)^{1/2}} dy \\ &= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y - x|^2)^{1/2}} dy \end{aligned}$$

Hence

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y - x|^2)^{1/2}} dy\right) + \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{(t^2 - |y - x|^2)^{1/2}} dy \tag{0.8}$$

Remark 6.0.2 *1. From the formula* (0.8) *it is clear that the range of influence is the interior of the cone $\{(x,t) : |x - x_0| \le ct\}$. Also the domain of dependence is the interior of disc $\{x + ct\xi, |\xi| \le 1\}$. So the disturbance at x_1 is felt after the time $t = \frac{|x_1|}{c}$ and it persists for all times $t > \frac{|x_1|}{c}$. This is also called absence of sharp signals. For example drop a stone in the water at x. The waves will be moving and is felt at a point x_1 at time t and later.*

2. *Loss of regularity: The formula* (0.8) *implies that if* $g \in C^{k+1}, h \in C^k$*, then* $u \in C^k$*. That is there is a loss of regularity.*

6.0.2 Non-homogeneous Problems: (Duhamel's Principle)

In this section we will see closed form solution like D'alemberts type for non-homogeneous wave equation:

$$u_{tt} - c^2 \Delta u = f(x,t), x \in \mathbb{R}^n, t > 0 \tag{0.9}$$

$$u(x,0) = 0,\ u_t(x,0) = 0,\ x \in \mathbb{R}^n. \tag{0.10}$$

Recall the Duhamel's principle for system of ODE's

$$X' + AX = F(t),\ X(0) = X_0$$

Then

$$X(t) = X_0 e^{At} + \int_0^t e^{-A(t-s)} F(s) ds$$

If we define $S(t)$ as solution operator in the sense that $X_0 \mapsto S(t)X_0$ satisfies

$$X' + AX = 0,\ X(0) = X_0$$

and the above formula is actually written as

$$X(x) = X_0 e^{at} + \int_0^t S(t-s) F(s) ds$$

So if we can define $S(t)$ for wave equation then we can write a candidate for solution. The above wave equation may be written as the system:

$$u_t = v,$$
$$v_t = u_{tt} = c^2 \Delta u + f(x,t)$$

This may be written as first order system with $U = (u,v)^T$

$$U_t + AU = F,\ U(0) = (0,0)^T$$

where $A = \begin{pmatrix} 0 & 1 \\ c^2\Delta & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}$.
So the solution of this problem is

$$U(t,x) = \int_0^t S(t-s) F(s,x) ds$$

where $S(t)$ is the solution operator of homogeneous system in the sense that for $\Phi = (\phi, \psi)^T$, $V(t,x) = S(t)\Phi$ satisfies

$$V_t + AV = 0,\ \ V(0) = \Phi$$

and $U(t,s,x) = S(t-s)F(x,s)$ is a solution of

$$U_t + AU = 0,\; U(t,t,s) = F(x,s) \text{on } \{t = s\}$$

That is

$$\begin{pmatrix} u_t \\ u_{tt} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ c^2\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} = 0,$$

and the initial condition

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ f(x,s) \end{pmatrix} \text{on } \{t = s\}$$

So it is equivalent to solve the initial value problem

$$u_{tt} - c^2\Delta u = 0, x \in \mathbb{R}^n, t > s \tag{0.11}$$

$$u(x,0) = 0,\; u_t(x,s) = f(x,s),\; x \in \mathbb{R}^n,\; \{t = s\}. \tag{0.12}$$

In dimension $n = 1$ this is given by the D'Alemberts formula

$$u(x,t,s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$$

Hence the candidate for solution of non-homogenous equation is

$$u(x,t) = \int_0^t u(x,t,s)ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\; dy\; ds \tag{0.13}$$

Using the fundamental theorem, one can show the following theorem:

Theorem 6.0.2 *If $U(x,t,s)$ is C^2 in x and t, continuous in s, and solves the problem* (0.11)-(0.12)*, then*

$$u(x,t) = \int_0^t U(x,t,s)ds$$

solve the problem (0.9)-(0.10).

6.0.3 problems

1. Show that if $u(x,t)$ satisfies parallologram property and is sufficiently smooth then $u(x,t)$ is a classical solution of the wave equation.
2. Show that (0.13) satisfies the non-homogeneous problem (0.9)-(0.10).