



ME 582 Advanced Materials Science

Chapter 3 Micromechanical Analysis of a Lamina (Part 2)

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Ultimate Strengths of a Unidirectional Lamina

- ☐ Longitudinal tensile strength $(\sigma_1^T)_{ult}$
- ☐ Longitudinal compressive strength $(\sigma_1^C)_{ult}$
- ☐ Transverse tensile strength $(\sigma_2^T)_{ult}$
- ☐ Transverse compressive strength $(\sigma_2^C)_{ult}$
- ☐ In-plane shear strength $(\tau_{12})_{ult}$



Longitudinal Tensile Strength

Assumptions (Mechanics of Materials Approach)

- ❑ Fiber and matrix are isotropic, homogeneous, and linearly elastic until failure
- ❑ The failure strain for the matrix is higher than for the fiber.
- ❑ When the fibers fail at the strain of $(\epsilon_f)_{ult}$, the whole composite fails

$$(\sigma_1^T)_{ult} = (\sigma_f)_{ult} V_f + (\epsilon_f)_{ult} E_m (1 - V_f)$$



Stress-Strain Curve for Composite

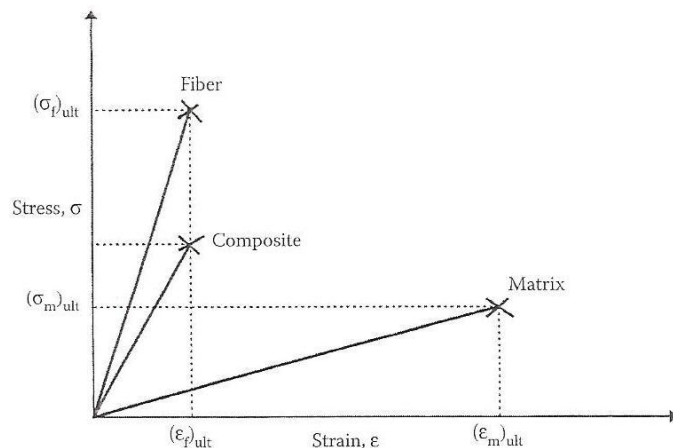


FIGURE 3.24

Stress-strain curve for a unidirectional composite under uniaxial tensile load along fibers.



Example

Example: Estimate the delivered fiber strength of the glass fiber in a glass/epoxy composite with a fiber volume fraction of 55 percent

- Composite: tensile strength = 56 ksi, failure strain = 0.010 (1%)
- Epoxy: tensile strength = 7 ksi, failure strain = 0.015 (1.5%), $E = 0.60 \times 10^6$ psi

$$\begin{aligned}\sigma_{lt} &= \sigma_{f \max} V_f + (\sigma_m)_{\epsilon_{f \max}} V_m \\ (\sigma_m)_{\epsilon_{f \max}} &= E_m \epsilon_{f \max} = (0.60 \times 10^6 \text{ psi}) (0.010) = 6.0 \text{ ksi} \\ 56 \text{ ksi} &= \sigma_{f \max} (0.55) + (6.0 \text{ ksi}) (.45) \\ \sigma_{f \max} &= 97 \text{ ksi}\end{aligned}$$

Thus, a composite tensile strength of 56 ksi translates into a delivered fiber strength of 97 ksi.



Critical Fiber Volume Fraction

It is also possible that, by adding fibers to the matrix, the composite will have lower ultimate tensile strength than the matrix. In that case, the fiber volume fraction for which this is possible is called the critical fiber volume fraction, $(V_f)_{critical}$, and is

$$\begin{aligned}(\sigma_m)_{ult} &> (\sigma_f)_{ult} (V_f)_{critical} + (\epsilon_f)_{ult} E_m [1 - (V_f)_{critical}] \\ (V_f)_{critical} &< \frac{(\sigma_m)_{ult} - E_m (\epsilon_f)_{ult}}{(\sigma_f)_{ult} - E_m (\epsilon_f)_{ult}}.\end{aligned}\tag{3.166}$$



Experimental Evaluation

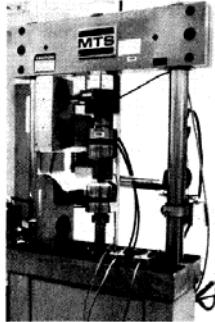


FIGURE 3.25
Tensile coupon mounted in the test frame for finding the tensile strengths of a unidirectional lamina. (Photo courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH.)

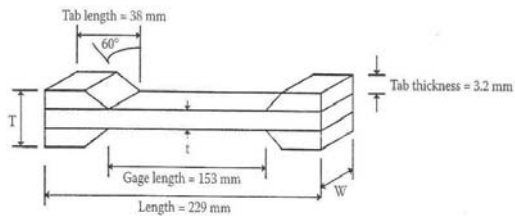


FIGURE 3.26
Geometry of a longitudinal tensile strength specimen.



Stress-Strain Curve for a Laminate

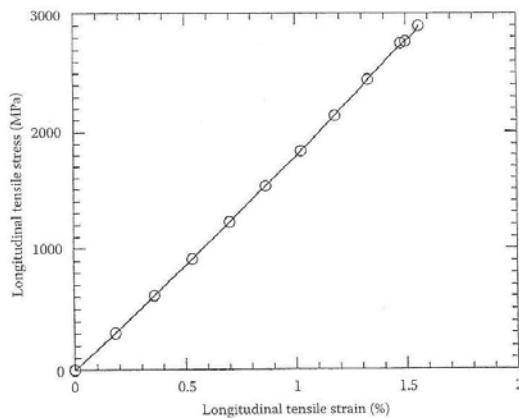


FIGURE 3.27
Stress-strain curve for a $[0]_s$ laminate under a longitudinal tensile load. (Data courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH).

$$E_1 = 187.5 \text{ GPa}$$

$$(\sigma_1^T)_{ult} = 2896 \text{ MPa}$$

$$(\epsilon_1^T)_{ult} = 1.560\%$$



Failure Under Longitudinal Tensile Load

- ☐ Brittle fracture of fibers
- ☐ Brittle fracture of fibers with pullout
- ☐ Fiber pullout with fiber-matrix debonding



Modes of Failure (Longitudinal Tensile Load)

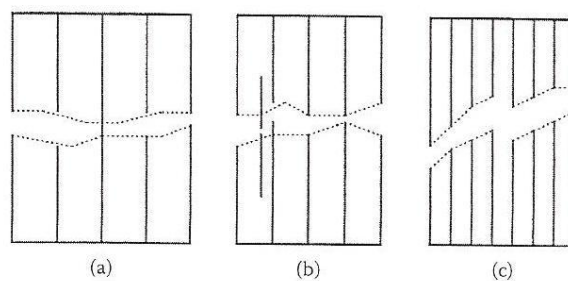


FIGURE 3.28

Modes of failure of unidirectional lamina under a longitudinal tensile load.

The mode of failure depends on the fiber-matrix bond strength and fiber volume fraction

- ☐ Low fiber volume fractions ($< 40\%$), model (1) failure
- ☐ Intermediate fiber volume fractions ($40\text{--}65\%$), model (2) failure
- ☐ High fiber volume fractions ($> 65\%$), model (3) failure



Failure Under Longitudinal Compressive Load

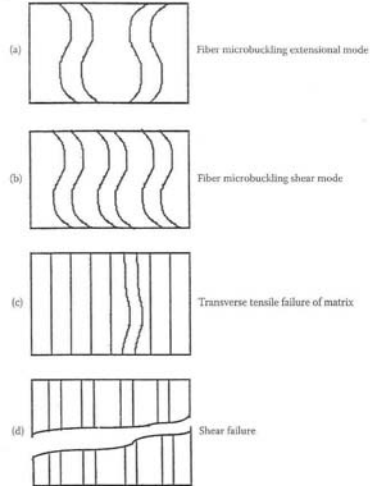


FIGURE 3.29 Modes of failure of a unidirectional lamina under a longitudinal compressive load.

- ☐ Fracture of matrix and/or fiber-matrix bond due to tensile strains in the matrix and/or bond
- ☐ Microbuckling of fibers in shear or extensional mode
- ☐ Shear failure of fibers



Matrix Failure Mode

$$|\varepsilon_1| = \frac{|\sigma_1|}{E_1}$$

$$|\varepsilon_2| = \nu_{12} |\varepsilon_1| = \nu_{12} \frac{|\sigma_1|}{E_1}$$

$$|\sigma_1| = \frac{E_1 |\varepsilon_2|}{\nu_{12}}$$

Maximum Strain Failure Theory

- ☐ If the transverse strain exceeds the ultimate transverse tensile strain, $(\varepsilon_2^T)_{ult}$, the lamina is considered to have failed in the transverse direction

$$(\sigma_1^c)_{ult} = \frac{E_1 (\varepsilon_2^T)_{ult}}{\nu_{12}}$$



Ultimate Transverse Tensile Strain

$$(\sigma_1^c)_{ult} = \frac{E_1 (\varepsilon_2^T)_{ult}}{\nu_{12}}$$

□ Empirical Formula

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult} (1 - V_f^{1/3})$$

□ Mechanics of Materials Formula

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult} \left[\frac{d}{s} \left(\frac{E_m}{E_f} - 1 \right) + 1 \right]$$



Shear/Extensional Fiber Microbuckling Failure

Shear/extensional fiber microbuckling failure mode: local buckling models for calculating longitudinal compressive strengths have been developed.^{21,22} Because these results are based on advanced topics, only the final expressions are given:

$$(\sigma_1^c)_{ult} = \min[S_1^c, S_2^c], \quad (3.172)$$

where

$$S_1^c = 2 \left[V_f + (1 - V_f) \frac{E_m}{E_f} \right] \sqrt{\frac{V_f E_m E_f}{3(1 - V_f)}}, \quad (3.173a)$$

and

$$S_2^c = \frac{G_m}{1 - V_f}. \quad (3.173b)$$



Shear Stress Failure of Fibers Mode

- A unidirectional composite may fail due to direct shear failure of fibers (rule of mixture).

$$(\tau_{12})_{ult} = (\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m$$

- Maximum shear stress

$$(\tau_{12})_{ult} = \frac{1}{2}(\sigma_1^C)_{ult}$$

$$(\sigma_1^C)_{ult} = 2(\tau_{12})_{ult} = 2[(\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m]$$



Factors to Affect the Predicted Values

- ❑ Irregular spacing of fibers causing premature failure in matrix-rich areas
- ❑ Less than perfect bonding between the fiber and the matrix
- ❑ Poor alignment of fibers
- ❑ Not accounting for Poisson's ratio mismatch between the fiber and the matrix
- ❑ Not accounting for the transversely isotropic nature of fibers such as aramids and graphite



Experimental Evaluation

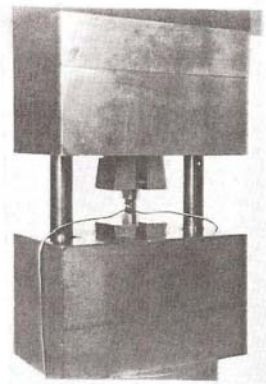
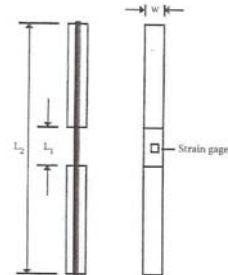


FIGURE 3.30
IITRI fixture mounted in a test frame for finding the compressive strengths of a lamina. (Data reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 76. Copyright CRC Press, Boca Raton, FL.)



Specimen dimensions		
L_1 , mm	L_2 , mm	w , mm
12.7 ± 1	12.7 ± 1.5	12.7 ± 0.1 or 6.4 ± 0.1

FIGURE 3.31
Geometry of a longitudinal compressive strength specimen. (Data reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 76. Copyright CRC Press, Boca Raton, FL.)



Transverse Tensile Strength

Assumptions (Mechanics of Materials Approach)

- ☐ A perfect fiber-matrix bond
- ☐ Uniform spacing of fibers
- ☐ The fiber and matrix follow Hooke's law
- ☐ There are no residual stress



Representative Volume Element (RVE)

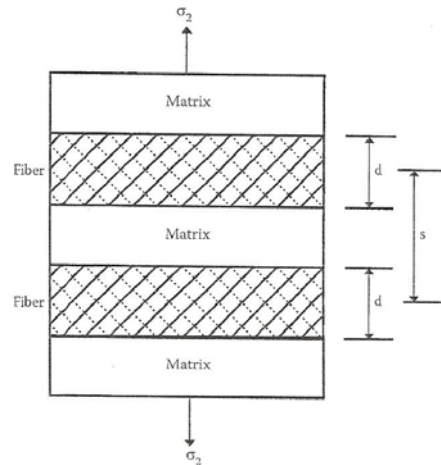


FIGURE 3.33
Representative volume element to calculate transverse tensile strength of a unidirectional lamina.



Transverse Tensile Strength

Now, by the definition of strain, the deformations are related to the transverse strains,

$$\delta_c = s\epsilon_c, \quad (3.177a)$$

$$\delta_f = d\epsilon_f, \quad (3.177b)$$

$$\delta_m = (s-d)\epsilon_m, \quad (3.177c)$$

where $\epsilon_{c,f,m}$ = the transverse strain in the composite, fiber, and matrix, respectively.

Substituting the expressions in Equation (3.82) in Equation (3.81), we get

$$\epsilon_c = \frac{d}{s}\epsilon_f + \left(1 - \frac{d}{s}\right)\epsilon_m. \quad (3.178)$$



Transverse Tensile Strength

$$E_f \epsilon_f = E_m \epsilon_m \quad (3.179)$$

Substituting the expression for the transverse strain in the fiber, i.e., in Equation (3.178), the transverse strain in the composite

$$\epsilon_c = \left[\frac{d}{s} \frac{E_m}{E_f} + \left(1 - \frac{d}{s} \right) \right] \epsilon_m \quad (3.180)$$

If one assumes that the transverse failure of the lamina is due to the failure of the matrix, then the ultimate transverse failure strain is

$$(\epsilon_c^T)_{ult} = \left[\frac{d}{s} \frac{E_m}{E_f} + \left(1 - \frac{d}{s} \right) \right] (\epsilon_m^T)_{ult} \quad (3.181)$$

where $(\epsilon_m^T)_{ult}$ = ultimate tensile failure strain of the matrix.

The ultimate transverse tensile strength is then given by

$$(\sigma_2^T)_{ult} = E_2 (\epsilon_c^T)_{ult} \quad (3.182)$$



Transverse Compressive Strength

3.4.4 Transverse Compressive Strength

Equation (3.182), which was developed for evaluating transverse tensile strength, can be used to find the transverse compressive strengths of a lamina. The actual compressive strength is again lower due to imperfect fiber/matrix interfacial bond and longitudinal fiber splitting. Using compressive parameters in Equation (3.182),

$$(\sigma_2^C)_{ult} = E_2 (\epsilon_c^C)_{ult} \quad (3.183)$$

where

$$(\epsilon_c^C)_{ult} = \left[\frac{d}{s} \frac{E_m}{E_f} + \left(1 - \frac{d}{s} \right) \right] (\epsilon_m^C)_{ult} \quad (3.184)$$

$(\epsilon_m^C)_{ult}$ = ultimate compressive failure strain of matrix.



In-Plane Shear Strength

$$\Delta_c = \Delta_f + \Delta_m. \quad (3.185)$$

By definition of shearing strain,

$$\Delta_c = s(\gamma_{12})_c, \quad (3.186a)$$

$$\Delta_f = d(\gamma_{12})_f, \quad (3.186b)$$

and

$$\Delta_m = (s-d)(\gamma_{12})_m, \quad (3.186c)$$

where $(\gamma_{12})_{c,f,m}$ = the in-plane shearing strains in the composite, fiber, and matrix, respectively.

Substituting the Equation (3.186a-c) in Equation (3.185),

$$(\gamma_{12})_c = \frac{d}{s}(\gamma_{12})_f + \left(1 - \frac{d}{s}\right)(\gamma_{12})_m. \quad (3.187)$$



In-Plane Shear Strength

$$(\gamma_{12})_m G_m = (\gamma_{12})_f G_f. \quad (3.188)$$

Substituting the expression for $(\gamma_{12})_f$ from Equation (3.188) in Equation (3.187),

$$(\gamma_{12})_c = \left[\frac{d}{s} \frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right) \right] (\gamma_{12})_m. \quad (3.189)$$

If one assumes that the shear failure is due to failure of the matrix, then

$$(\gamma_{12})_{ult} = \left[\frac{d}{s} \frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right) \right] (\gamma_{12})_{m,ult}, \quad (3.190)$$

where $(\gamma_{12})_{m,ult}$ = ultimate shearing strain of the matrix.

The ultimate shear strength is then given by

$$\begin{aligned} (\tau_{12})_{ult} &= G_{12}(\gamma_{12})_{ult} \\ &= G_{12} \left[\frac{d}{s} \frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right) \right] (\gamma_{12})_{m,ult}. \end{aligned} \quad (3.191)$$



Longitudinal Thermal Expansion Coefficient

the longitudinal direction under a temperature change of ΔT . If only the temperature ΔT is applied, the unidirectional lamina has zero overall load, F_1 , in the longitudinal direction. Then

$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m \quad (3.196)$$

$$\sigma_f V_f + \sigma_m V_m = 0, \quad (3.197)$$

where

$A_{c,f,m}$ = the cross-sectional area of composite, fiber, and matrix, respectively

$\sigma_{1,f,m}$ = the stress in composite, fiber, and matrix, respectively

Although the overall load in the longitudinal direction 1 is zero, stresses are caused in the fiber and the matrix by the thermal expansion mismatch between the fiber and the matrix. These stresses are

$$\sigma_f = E_f(\epsilon_f - \alpha_f \Delta T), \quad (3.198a)$$

$$\sigma_m = E_m(\epsilon_m - \alpha_m \Delta T). \quad (3.198b)$$



Longitudinal Thermal Expansion Coefficient

$$\epsilon_f = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m} \Delta T. \quad (3.199)$$

For free expansion in the composite in the longitudinal direction 1 the longitudinal strain is

$$\epsilon_1 = \alpha_1 \Delta T. \quad (3.200)$$

Because the strains in the fiber and composite are also equal ($\epsilon_1 = \epsilon_f$), from Equation (3.199) and Equation (3.200),

$$\alpha_1 = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m}.$$

Using Equation (3.34) for the definition of longitudinal Young's modulus,

$$\alpha_1 = \frac{1}{E_1} (\alpha_f E_f V_f + \alpha_m E_m V_m). \quad (3.201)$$

The longitudinal coefficient of thermal expansion can be rewritten as

$$\alpha_1 = \left(\frac{\alpha_f E_f}{E_1} \right) V_f + \left(\frac{\alpha_m E_m}{E_1} \right) V_m, \quad (3.202)$$

which shows that it also follows the rule of mixtures based on the weighted mean of $\alpha E/E_1$ of the constituents.

- Rule of mixtures approach
- Predictions agree well with experimental data



Transverse Thermal Expansion Coefficient

Due to temperature change, ΔT , assume that the compatibility condition that the strain in the fiber and matrix is equal in direction 1 — that is,

$$\epsilon_m = \epsilon_f = \epsilon_1. \quad (3.203)$$

Now, the stress in the fiber in the longitudinal direction 1 is

$$\begin{aligned} (\sigma_f)_1 &= E_f (\epsilon_f)_1 \\ &= E_f \epsilon_1 \\ &= E_f (\alpha_1 - \alpha_f) \Delta T \end{aligned} \quad (3.204)$$

and the stress in the matrix in longitudinal direction 1 is

$$\begin{aligned} (\sigma_m)_1 &= E_m (\epsilon_m)_1 \\ &= E_m \epsilon_1 \\ &= -E_m (\alpha_m - \alpha_1) \Delta T. \end{aligned} \quad (3.205)$$



Transverse Thermal Expansion Coefficient

$$(\epsilon_f)_2 = \alpha_f \Delta T - \frac{\nu_f (\sigma_f)_1}{E_f} \quad (3.206)$$

$$(\epsilon_m)_2 = \alpha_m \Delta T - \frac{\nu_m (\sigma_m)_1}{E_m}. \quad (3.207)$$

The transverse strain of the composite is given by the rule of mixtures as

$$\epsilon_2 = (\epsilon_f)_2 V_f + (\epsilon_m)_2 V_m. \quad (3.208)$$

Substituting Equation (3.206) and Equation (3.207) in Equation (3.208),

$$\begin{aligned} \epsilon_2 &= \left[\alpha_f \Delta T - \frac{\nu_f E_f (\alpha_1 - \alpha_f) \Delta T}{E_f} \right] V_f \\ &\quad + \left[\alpha_m \Delta T + \frac{\nu_m E_m (\alpha_m - \alpha_1) \Delta T}{E_m} \right] V_m \end{aligned} \quad (3.209)$$



Transverse Thermal Expansion Coefficient

$$\epsilon_2 = \alpha_2 \Delta T, \quad (3.210)$$

we get

$$\alpha_2 = [\alpha_f - \nu_f (\alpha_1 - \alpha_f)] V_f + [\alpha_m + \nu_m (\alpha_1 - \alpha_f)] V_m. \quad (3.211)$$

Substituting

$$\nu_{12} = \nu_f V_f + \nu_m V_m \quad (3.212)$$

in the preceding equation, it can be rewritten as

$$\alpha_2 = (1 + \nu_f) \alpha_f V_f + (1 + \nu_m) \alpha_m V_m - \alpha_1 \nu_{12}. \quad (3.213)$$



Longitudinal Coefficients of Moisture Expansion

$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m, \text{ and}$$

$$\sigma_f V_f + \sigma_m V_m = 0, \quad (3.220)$$

where

$A_{f,m}$ = the cross-sectional areas of the fiber, matrix, and composite, respectively

$\sigma_{f,m}$ = the stresses in the fiber, matrix, and composite, respectively

The stresses in the fiber and matrix caused by moisture are

$$\sigma_f = E_f (\epsilon_f - \beta_f \Delta C_f), \quad (3.221)$$

$$\sigma_m = E_m (\epsilon_m - \beta_m \Delta C_m). \quad (3.222)$$

Substituting Equation (3.221) and (3.222) in Equation (3.220) and knowing that the strains in the fiber and matrix are equal ($\epsilon_f = \epsilon_m$),

$$\epsilon_f = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_f V_f + E_m V_m}. \quad (3.223)$$



Longitudinal Coefficients of Moisture Expansion

$$\epsilon_1 = \beta_1 \Delta C_c, \quad (3.224)$$

where ΔC_c = the moisture concentration in composite.

Because the strains in the fiber and the matrix are equal,

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m) \Delta C_c}. \quad (3.225)$$

Equation (3.225) can be simplified by relating the moisture concentration in the composite (ΔC_c) to the moisture concentration in the fiber (ΔC_f) and the matrix (ΔC_m).

The moisture content in the composite is the sum of the moisture contents in the fiber and the matrix,

$$\Delta C_c w_c = \Delta C_f w_f + \Delta C_m w_m, \quad (3.226)$$



Longitudinal Coefficients of Moisture Expansion

$$\Delta C_c = \Delta C_f W_f + \Delta C_m W_m, \quad (3.227)$$

where $W_{f,m}$ = the mass fractions of the fiber and matrix, respectively.

Substituting Equation (3.227) in Equation (3.225),

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m)(\Delta C_f W_f + \Delta C_m W_m)}. \quad (3.228)$$

Using Equation (3.4) and Equation (3.34), one can rewrite Equation (3.228) in terms of fiber volume fractions and the longitudinal Young's modulus as

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_1 (\Delta C_f \rho_f V_f + \Delta C_m \rho_m V_m)} \rho_c. \quad (3.229)$$



Transverse Coefficients of Moisture Expansion

$$\beta_2 = \frac{V_f(1 + v_f)\Delta C_f\beta_f + V_m(1 + v_m)\Delta C_m\beta_m}{(V_m\rho_m\Delta C_m + V_f\rho_f\Delta C_f)}\rho_c - \beta_1v_{12}, \quad (3.215)$$

where

ΔC_f = the moisture concentration in the fiber, kg/kg (lb/lb)

ΔC_m = the moisture concentration in the matrix, kg/kg (lb/lb)

β_f = the coefficient of moisture expansion of the fiber, m/m/kg/kg (in./in./lb/lb)

β_m = the coefficient of moisture expansion of the matrix, m/m/kg/kg (in./in./lb/lb)