ELL225

Lecture-11

For given $G(s)$ $\longrightarrow$ How to obtain time response
& input u

$\downarrow$

Using Partial fraction method

$$\left\{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}\right\} \rightarrow \text{Assume that } x(0) = 0$$

Laplace transform

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = CX(s)$$
$$= C(sI-A)^{-1}BU(s)$$

$$\Rightarrow \boxed{\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B} \leftarrow G(s)$$

$\hookrightarrow$ Consider a system whose dynamics is

$$\left\{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}\right.$$

What is the time response of the system $\rightarrow$ Obtain $x(t)$

the output $y = Cx(t)$

One may consider following procedure

The procedure is correct, provided the initial condition $x(0)$ is set to 0
$x(0)=0$

→ For a given state space representation, compute the associated transfer function

↓

then use partial fraction method (with Laplace inverse) to compute $y(t)$

↳ If we assume that $x(0) \neq 0$, then above approach will NOT work!

↓

First consider an autonomous system

$x \in \mathbb{R}^n$, A is of size $n \times n$

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

$$\dot{x} = \alpha x \qquad x(0) = x_0 \qquad x \in \mathbb{R}$$

soln

$$x(t) = e^{\alpha t} x_0 \quad \checkmark$$

$$e^{\alpha t} = 1 + \alpha t + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \cdots$$

$$x(t) = \left[1 + \alpha t + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \cdots \right] x_0 \quad \text{---} \quad (1)$$

Let us define

$$x \in \mathbb{R}^n \rightarrow \bar{x}(t) := \underbrace{\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right]}_{e^{At}} x_0$$

$$\dot{\bar{x}}(t) := \left[0 + A + A^2 t + \frac{1}{2!}A^3t^2 + \cdots + \right] x_0$$

$$= A \underbrace{\left[I + At + \frac{1}{2!}A^2t^2 + \cdots\cdots\right] x_0}_{\bar{x}}$$

$$= A\bar{x}$$

This implies $\bar{x}(t)$ is a solution $\dot{x} = Ax$

$$e^{At} := I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots.$$

$\dot{x} = Ax \qquad x(0) = x_0$

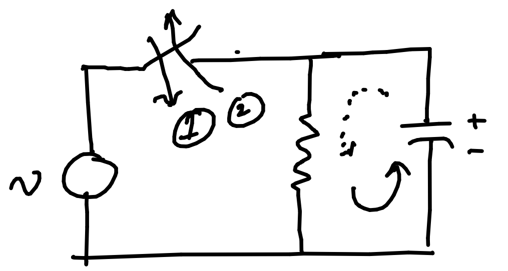
soln $\left(\boxed{x(t) = e^{At} x_0} \right. \rightarrow y(t) = C x(t)$

$= C e^{At} x_0$

→ When there is no external input to the system, the response (output) completely depends on the initial condition of the system

response of the syst

→ If $x(0) = 0$ then $x(t) = 0$



switch was closed

for some time

then opened

The capacitor has charged when

the switch was closed (Case 1)

↳ when the switch is opened (Case 2)

there will be current flow in the

ckt through resistor.

⇓

the current is not zero, which is

due to the precharged capacitor

(non-zero initial

cond^n)

→ Since there is no input to the system

the response $\underline{\underline{y(t) = Ce^{At}x_0}}$ is called

"Natural response" of the system.

"Zero-input response" of the syst..

↳

$$\dot{x} = Ax \qquad x(0) = x_0$$

Laplace transform

$$sX(s) - x_0 = AX(s)$$

$$\Rightarrow (sI - A)X(s) = x_0$$

$$\Rightarrow X(s) = (sI - A)^{-1}x_0$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)x_0$$

$$x(t) = \underbrace{\mathcal{L}^{-1}\left[(sI - A)^{-1}\right]}_{\phi(t) = e^{At}} x_0 \quad \cdots$$

<u>Ex</u>

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\downarrow$$

$$x(t) = e^{At} x_0$$

$$e^{At} = \left[I + At + \frac{1}{2} A^2 t^2 + \cdots \right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} + \cdots$$

$$= \begin{bmatrix} 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \cdots & 0 \\ 0 + t - \frac{3t^2}{2!} + \frac{7t^3}{3!} + \cdots & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} \\ e^{-t} - e^{-2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$$

Usig Laplace transfm approach:

$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] x_0$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} = e^{At}$$

$$\boxed{e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]}$$

$$y(t) = C x(t) = C \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= C \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

Now consider a non-autonomous system
(forced system)

$$\dot{x} = Ax + Bu \quad \text{with } x(0) = x_0$$

$$y = Cx$$

external force / input

taking Laplace transform

$$sX(s) - x_0 = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = x_0 + BU(s)$$

$$\Rightarrow X(s) = [sI - A]^{-1}x_0 + [sI - A]^{-1}BU(s)$$

for $t > 0$

$$x(t) = \underbrace{\mathcal{L}^{-1}[(sI - A)^{-1}]x_0}_{e^{At}x_0} + \mathcal{L}^{-1}[(sI - A)^{-1}BU(s)]$$

$$= e^{At}x_0 + \mathcal{L}^{-1}[(sI - A)^{-1}] * \mathcal{L}^{-1}(BU(s))$$

convolution operation

$$= e^{At}x_0 + e^{At} * Bu(t)$$

$$= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\boxed{x(t) = \underbrace{e^{At}x_0}_{\text{zero input or Natural response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{zero state response (assuming the initial cond. } x_0 = 0)}} \quad t > 0$$

Take $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$

Assume that the input $u(t)$ is a <u>unit step func^n</u>.

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$\underline{\underline{f(t)\ u(t)}} = \begin{cases} f(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$e^{At} x_0$ already computed.

Let us now consider $x_0 = 0$

(One can obtain its transfer funct^n)

$$x(t) = e^{At} \int_0^t e^{-A\tau} B u(\tau)\, d\tau$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{2\tau} & 0 \\ e^{\tau} - e^{2\tau} & e^{\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} \int_0^t e^{2\tau}\, d\tau \\ \int_0^t e^{\tau} - e^{2\tau}\, d\tau \end{bmatrix}$$

= Complete it ? !

Do the following & verify

- Obtain transfunct
- Perform partial fraction
- Take Laplace inverse & compute the time response

compare