ELL225

Lecture-12

For given system
with transfer funct^n
$G(s)$ & ✓

State space realization $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ ✓

$$G(s) = C(sI - A)^{-1}B$$ ✓

↓

To obtain time response

→ By using partial fraction method (for $G(s)$)

→ By using e^{At} evaluation (for state space representat^n)

$$\dot{x} = Ax$$

$$x(t) = e^{At}x_0 \equiv \phi(t)\, x_0$$

↑ state transition matrix

The state response $x(t)$ is completely determined by $\phi(t)$. $\phi(t)$ is a map from initial condition x_0 to $x(t)$.

Difficulties with the procedures followed.

① Partial fraction method

Difficulties:

(i) One has to assume that the initial condition is zero

(ii) One assumes that the roots of polynomial $a(s)$ $\left(G(s) = \frac{b(s)}{a(s)}\right)$ are known or can be computed, which is not an easy task for higher degree polynomials.

To address the above difficulties, particularly due to the presence of non-zero initial condition, we use following state-space approach:

② Computation of state transition matrix

$$\Phi(t) := \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = e^{At} \qquad t \geq 0$$

→ Difficulties:

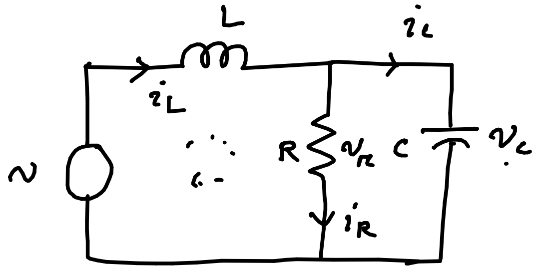
For a large scale system, the computation of $\Phi(t)$ is not so easy.

$$\Phi(t) = \mathcal{L}^{-1}\left((sI-A)^{-1}\right)$$

→ Any other method by which the computation of $\mathcal{L}^{-1}\left((sI-A)^{-1}\right)$ can be avoided.

→ New methodology :

Consider a ckt. example :



→ Let us choose following quantities as state variables :

- inductor current i_L
- capacitor voltage v_c

Apply KVL

$$L\frac{di_L}{dt} = -v_c + v \quad \text{—— (1)}$$

Applying KCL $\quad i_L = i_R + i_c$

$$\Rightarrow i_L = \frac{v_c}{R} + C\frac{dv_c}{dt} \quad \text{—— (2)}$$

Combining (1) & (2), we have :

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} v \leftarrow \text{S.S.R - 1}$$

state-space representation 1

↳ Consider the same circuit & choose following quantities as state variables :

- inductor current i_L
- resistor current i_R

$$\frac{di_L}{dt} = -\frac{R}{L} i_R + \frac{1}{L} v \qquad \cdots \text{(1)}$$

$$i_c = i_L - i_R \qquad\qquad v_c = v_R = i_R R$$

$$\frac{1}{C}\int i_c \, dt = i_R R$$

$$R \frac{di_R}{dt} = \frac{1}{C}(i_L - i_R)$$

$$\Rightarrow \frac{di_R}{dt} = \frac{1}{RC} i_L - \frac{1}{RC} i_R \qquad \cdots \text{(2)}$$

Combining (1) & (2), we have:

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{di_R}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{R}{L} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} i_L \\ i_R \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\tilde{B}} v \qquad \cdots \boxed{S.S.R-2}$$

→ The same circuit dynamics is represented by two different state-space representations.

- Any relation between state variables x & z?

$$\underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} \underset{Q^{-1}}{\overset{Q}{\rightleftarrows}} \underbrace{\begin{bmatrix} i_L \\ i_R \end{bmatrix}}_{z}$$

$$\begin{bmatrix} i_L \\ i_R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix}}_{Q \,\leftarrow\, \text{A non-singular matrix}} \begin{bmatrix} i_L \\ v_C \end{bmatrix} \quad \rightarrow \quad z = Qx$$

- The state variables are related through some non-singular transformation matrix Q.

Let us choose $L = 1$ $C = 1$ $R = 0.5$

Then S.S.R - 1 is :

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} v$$

S.S.R - 2 will be

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{di_R}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -0.5 \\ 2 & -2 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} i_L \\ i_R \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\tilde{B}} v$$

$$z = Qx \quad \Rightarrow x = Q^{-1} z$$

$$\text{SSR-1} \rightarrow \dot{x} = Ax + Bu$$

$$Q^{-1}\dot{z} = AQ^{-1}z + Bu$$

$$\Rightarrow \dot{z} = \underbrace{QAQ^{-1}}_{\tilde{A}} z + \underbrace{QB}_{\tilde{B}} u$$

S.S.R - 2

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 2 & -2 \end{bmatrix} = \tilde{A}$$

→ For a given system, one can find a different state space representation, depending upon the choice of state variables.

→ For a given system, assume that following is a state space representation:

$$\text{S.S.R-1} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

- Choose a different state variable z for the same system such that

$$\boxed{z = V^{-1}x} \qquad \boxed{x = Vz}$$

where V is a non-singular transformation matrix

$$V\dot{z} = AVz + Bu$$

$$\Rightarrow \dot{z} = \underbrace{V^{-1}AV}_{\tilde{A}} + \underbrace{V^{-1}B}_{\tilde{B}}u$$

$$y = \underbrace{CV}_{\tilde{C}}z$$

This process is called "SIMILARITY TRANSFORMATION"

$$\text{SSR-2} \begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z \end{cases} \rightarrow \begin{cases} \tilde{A} = V^{-1}AV \\ \tilde{B} = V^{-1}B \\ \tilde{C} = CV \end{cases}$$

S.S.R.-1 & S.S.R-2 are called "equivalent systems" or "similar systems".

→ S.S.R-1 and S.S.R-2 describe the dynamic behavior of same system. The difference is that for SSR-1, we have 'x' as state variable & for SSR-2, we have 'z' as state variable. The state variables x & z are related through some non-singular transformation matrix V.

Ref	State variables
SSR-1	x
SSR-2	z

$$x \xrightarrow{V^{-1}} z, \quad z \xrightarrow{V} x$$

$$z = V^{-1}x$$

→ How many state space representations are possible for a given system?

↓

" For a given system one can obtain infinitely many similar or equivalent state space representations, which are related through some Non-singular transformation matrix. "

* All the similar (equivalent) state space representations have SAME TRNSFER FUNCTION for a given system.

Prove