ELL 225

Lecture - 13

Similarity transformation

Given
$$\left\{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \text{SSR-1}$$

$$z = V^{-1}x$$

using non-singular transformation matrix V^{-1}

V

$$\left\{\begin{aligned} \dot{z} &= \tilde{A}z + \tilde{B}u \\ y &= \tilde{C}z \end{aligned}\right\} \text{SSR-2}$$

when $\tilde{A} = V^{-1}AV \qquad \tilde{B} = V^{-1}B \qquad \tilde{C} = CV$

- By choosing different Q one can obtain another state space representation of same system

- Transfer function :

$$G(s) = C(sI - A)^{-1}B \qquad \underline{\text{SSR - 1}}$$

$$\begin{aligned} \tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \\ &= CV(sI - V^{-1}AV)^{-1}V^{-1}B \\ &= CV(sV^{-1}V - V^{-1}AV)^{-1}V^{-1}B \\ &\;\vdots \quad \text{do this} \\ &\;? \\ &= C(sI - A)^{-1}B = G(s) \end{aligned}$$

- For a given system, the transfer function $G(s)$ is UNIQUE. However, it has multiple (equivalent / similar) state space representations.

↓

All the state space space realizations are related through some non-singular transformation matrix V.

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- All state space representations are NOT useful to us!

- One has to choose the transformation matrix V carefully to obtain some useful S.S.R.

→ Now we choose V to obtain time-response of an autonomous system

$$\dot{x} = Ax \quad , \quad x(0) = x_0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x(t) = ?$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\dot{z}_1 = -4z_1 \longrightarrow z_1(t) = e^{-4t} z_{10}$$

$$\dot{z}_2 = 2z_2 \longrightarrow z_2(t) = e^{2t} z_{20}$$

For a given system $\dot{x} = Ax \qquad x(0) = x_0$

Let us define another state variable

$$z = V^{-1}x \longrightarrow z(0) = V^{-1}x(0) \quad \cdots ①$$

$$V\dot{z} = AVz$$

$$\Rightarrow \dot{z} = V^{-1}AVz \qquad A \in \mathbb{R}^{n \times n}$$

↳ Assume that A has $\underline{n\text{-distinct}}$ $\underline{\text{eigenvalues}}$.

✓ $\{\lambda_1, \lambda_2, \cdots \lambda_n\}$

A has $\underline{n\text{-linearly independent}}$ eigenvectors.

$\{v_1, v_2 \cdots v_n\}$

Let us construct the non-singular transformation matrix

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}_{n \times n}$$

Non-singular matrix

$$\dot{z} = \underbrace{V^{-1}AV}\, z$$

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

diagonal matrix with diagonal entries are the n-distinct eigenvalues of matrix A.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

In the transformed system, computing the solution is easy.

$$\downarrow$$

$$z_i(t) = e^{\lambda_i t} z_i(0) \qquad \text{for } i = 1, 2 \cdots n$$

$$\underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} z_1(0) \\ z_2(0) \\ \vdots \\ z_n(0) \end{bmatrix}}_{z(0)}$$

We now have to go back to the solution of original system using the transformation matrix V.

$$z = e^{\Lambda t} z(0)$$

$$V^{-1}x = e^{\Lambda t} V^{-1} x(0)$$

$$\boxed{\Rightarrow x(t) = \underbrace{V e^{\Lambda t} V^{-1}}_{e^{At}} x(0)}$$

$$\dot{x} = Ax$$

$$\downarrow$$

$$x(t) = e^{At} x_0$$

$$W := V^{-1} \qquad W = \begin{bmatrix} - & w_1^T & - \\ - & w_2^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \underbrace{\begin{bmatrix} - & w_1^T & - \\ - & w_2^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix} \begin{bmatrix} | \\ x_0 \\ | \end{bmatrix}}_{w_i^T x_0}$$

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{x(t)} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \, w_1^T x_0 \\ e^{\lambda_2 t} \, w_2^T x_0 \\ \vdots \\ e^{\lambda_n t} \, w_n^T x_0 \end{bmatrix}$$

scalars

$$\boxed{x(t) = e^{\lambda_1 t}(w_1^T x_0)\, v_1 + e^{\lambda_2 t}(w_2^T x_0)\, v_2 + \cdots + e^{\lambda_n t}(w_n^T x_0)\, v_n}$$

The state transition matrix

$$\boxed{\varphi(t) = e^{At} = V e^{\Lambda t} V^{-1}}$$

$$\left\{ \begin{array}{l} v_i e^{\lambda_i t} \rightarrow i^{th} \text{ mode of a system} \\ \lambda_i \rightarrow i^{th} \text{ modal frequency} \end{array} \right.$$

$$\boxed{x(t) = V e^{\Lambda t} V^{-1} x_0}$$

<u>modal Decomposition</u> of sntn $\dot{x} = Ax$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Eigenvalues of $A = \overset{\lambda_1}{-4}, \overset{\lambda_2}{2}$

$\downarrow \quad \downarrow$

Eigenvectors $A = \underset{v_1}{\begin{bmatrix} 1 \\ -4 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$$[-4I - A]v_1 = 0$$

$$\begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = 0$$

Any vector v_i which satisfies

$$(\lambda_i I - A)v_i = 0$$

is an eigenvector of A corresponding to the eigenvalue λ_i.

$$[2I - A]v_2 = 0$$

$$\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}\right) v_2 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{v_2} = 0$$

Second step: form $V = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$

$$W = V^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = V e^{\Lambda t} V^{-1} x_0$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} V^{-1}$$

$$\frac{1}{-4} \quad \frac{1}{2}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} \\ 11e^{2t} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} e^{-4t} + 11e^{2t} \\ -4e^{-4t} + 22e^{2t} \end{bmatrix}$$

$$x_1(t) = \frac{1}{6} \left(e^{-4t} + 11e^{2t} \right)$$

$$x_2(t) = \frac{1}{6} \left(-4e^{-4t} + 22e^{2t} \right)$$