ELL 225

Lecture - 14

STABILITY ANALYSIS

System :
$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad x(0) = x_0$$

Definition-1

The system Σ is stable (assymptotically stable) if the natural response of the system approaches to 0 as $t \to \infty$

The soln of Σ :

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$y(t) = Cx(t) = \underbrace{Ce^{At}x_0}_{\substack{\text{Natural resp} \\ y_n(t)}} + \underbrace{C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\substack{\text{forced resp} \\ y_f(t)}}$$

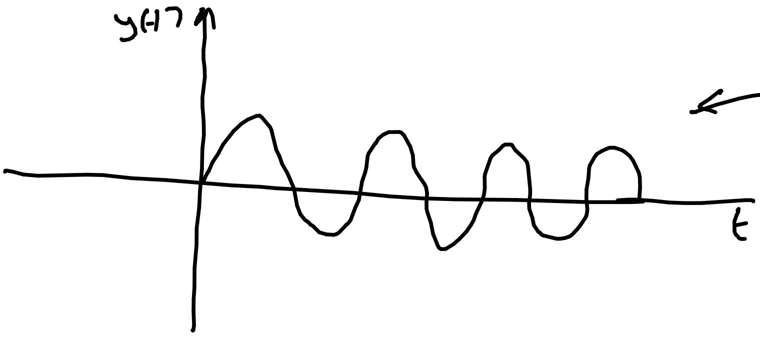
STABLE if $\lim_{t \to \infty} y_n(t) = 0$

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \qquad x(0) = x_0$$

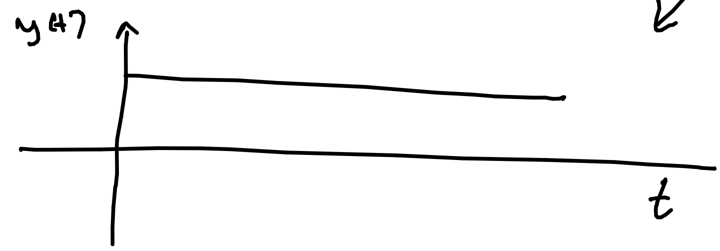
$$\to y(t) = Ce^{At}x_0$$

* A system is said to be <u>marginally stable</u> if the natural response $y_n(t)$ neither decays, nor grows, but remains constant or oscillate as $t \to \infty$.

← Stable syst.



← Marginally Stable



$$\dot{x} = Ax \qquad x(0) = x_0$$

→ If the response of autonomous system $(\dot{x} = Ax,\ y = Cx)$ goes to zero as $t \to \infty$ then the system is stable.

$$\dot{x} = Ax \qquad x(0) = x_0$$

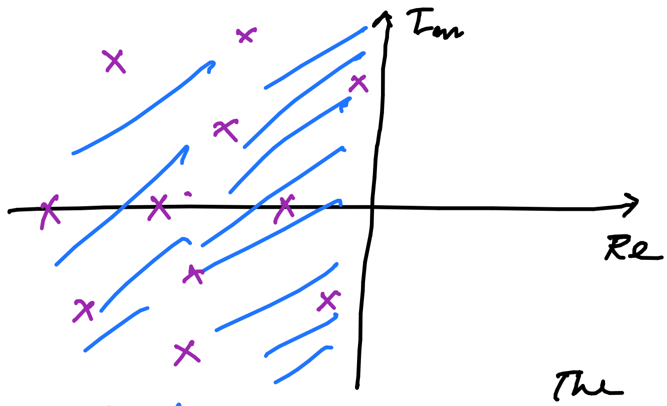
Let $\quad V := \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{\text{eigenvectors}} \qquad \underbrace{\lambda_1 \quad \lambda_2 \cdots \lambda_n}_{\substack{n\text{-distinct} \\ \text{eigenvalues}}}$

$$V^{-1} = W = \begin{bmatrix} - w_1^T - \\ - w_2^T - \\ \vdots \\ - w_n^T - \end{bmatrix}$$

$$x(t) = (w_1^T x_0) \underline{e^{\lambda_1 t}} v_1 + (w_2^T x_0) \underline{e^{\lambda_2 t}} v_2 + \cdots + (w_n^T x_0) \underline{e^{\lambda_n t}} v_n$$

$y = Cx(t)$

- $x(t)$ will converge to zero as $t \to \infty$ if and only if the real parts of λ_i are negative



$Re(\lambda_i) < 0$

The eigenvalues of matrix A are in the open left half of complex pl.

Imaginary axis is excluded

$\rightarrow \quad \lim_{t \to \infty} x(t) = 0$ if and only if the eigenvalues λ_i of A are in the open left half of complex plane.

$$\Updownarrow$$

$$\lim_{t \to \infty} y_n(t) = 0$$

$Re(\lambda_i) < 0$

If the LTI system is represented by its transfer function : $G(s)$.

→ Definition - 2

Let the system be represented by its transfer function $G(s)$. The system is said to be STABLE if the impulse response $g(t)$ of the system goes to zero as $t \to \infty$.

$$\lim_{t \to \infty} g(t) = 0$$

$g(t) = \mathcal{L}^{-1}(G(s))$

terms in the response of a syst.

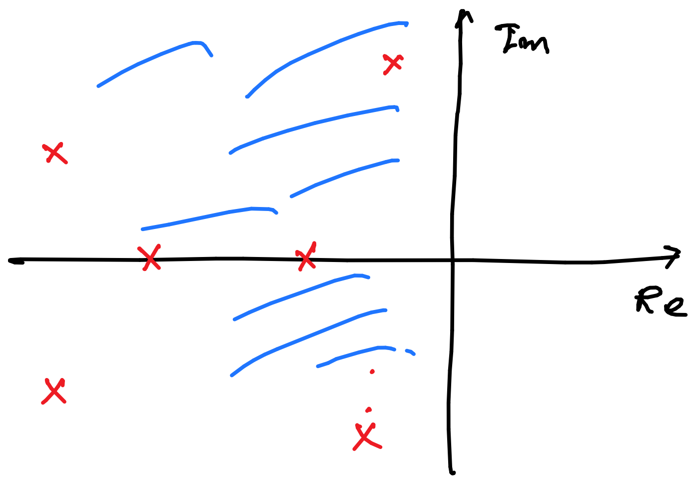
$$K e^{-\alpha t}, \quad K \frac{t^{r-1}}{(r-1)!} e^{-\alpha t}, \quad K e^{-\alpha t} \cos(\omega t - \varphi)$$

$$K \frac{t^{r-1}}{(r-1)!} e^{-\alpha t} \cos(\omega t - \varphi)$$

$g(t) \xrightarrow{t \to \infty} 0$

$G(s) = \dfrac{b(s)}{a(s)}$

$\lim_{t \to \infty} g(t) = 0$ if and only if the real parts of the roots of $a(s)$ are in open left half of complex plane



α_i are the roots of $a(s)$

$Re(\alpha_i) < 0$

→ There is a relation between t.f. & S.S.R.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$G(s) = C(sI - A)^{-1}B \nearrow g(t) = Ce^{At}B$$

→ For definition - 1 $\quad \lim_{t\to\infty} Ce^{At}x_0 = 0$

→ For defitin - 2 $\quad \lim_{t\to\infty} Ce^{At}B = 0$

↳ Assume that $G(s)$ is given & its poles are in open left half of Complex plane.

⇓

System is stable

↓

Can I say that the system $\dot{x} = Ax + Bu$, $y = Cx$ whose t.f. is $G(s)$ is also stable?

$$G(s) = \frac{b(s)}{a(s)} = c\,(sI-A)^{-1}b$$

$$= \frac{c\,\mathrm{adj}(sI-A)\,b}{\det(sI-A)}$$

$$\boxed{a(s) = \det(sI-A)}$$

* The roots of $a(s)$ are equal to the eigenvalues of A ?