ELL225

Lecture-18

Physical System $\xrightarrow{\text{linearize}}$ Given system

$$\left\{ \begin{array}{l} \dot{x} = Ax + bu \\ y = cx \end{array} \right\} \longrightarrow G(s)$$

on state space — Transfer funct°

↓

Time response analysis

↓

Stability Analysis

defined based on the time response of the systm

↓

To test if a system is

Stable / BIBO stable / Unstable

We need to know the position of the

eigenvalues of A or the poles of G(s).

$$G(s) = \frac{s+1}{s^4 + 3s^3 + 2s^2 + 3} \longleftarrow$$

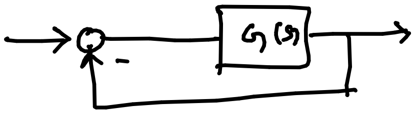
For stability checking we need to compute the roots.

↓

Is it easy?

For polynomial with degree ≥ 3, it is not so easy to compute the roots.

Even if G(s) is given in factored form, the closed loop characteristic polynomial becomes non-factored form.



$$G(s) = \frac{b(s)}{a(s)}$$

closed loop ch. polynomial $= a(s) + b(s)$

→ Routh-Hurwitz Criterion

Let say $$G(s) = \frac{1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$a(s)$

Routh Table :

$$\begin{array}{c|ccc}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & \dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & \dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_3 \\
s^1 & \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_2 & 0 \\
s^0 & \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1} & \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} &
\end{array}$$

- Even polynomial $a_e(s) = a_4 s^4 + a_2 s^2 + a_0$ — 1st row of Routh table
- Odd polynomial $a_o(s) = a_3 s^3 + a_1 s$ — 2nd row of Routh table.

$$s^3 + 10s^2 + 31s + 1030 \rightarrow$$

roots are:

-13.4136

$1.7068 \pm 8.595i$

	1st colum of Routh table		
s^3	1	31	0
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103	0
s^1	$\boxed{-72}$	0	0
s^0	103		

One sign chng

One sign change

There are 2 sign changes in the 1st colum. & here there are two roots in the right half of complex plane.

Routh - Hurwitz Criterion

" The number of roots of the polynomial that are in the right half of complex plane is equal to the number of sign changes in the 1st colum of Routh table "

- No. of roots in right half of $\mathbb{C}$ = the number of sign changes in 1st colum of Routh table

Special Cases

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

s^5	1	3	5
s^4	2	6	3
s^3	$\boxed{0} \rightarrow \epsilon$	$7/2$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3		

ϵ is a small number which may be +ve or −ve

Roots are:

$0.3429 \pm 1.5083i$

-1.6681

$-0.5088 \pm 0.7020i$

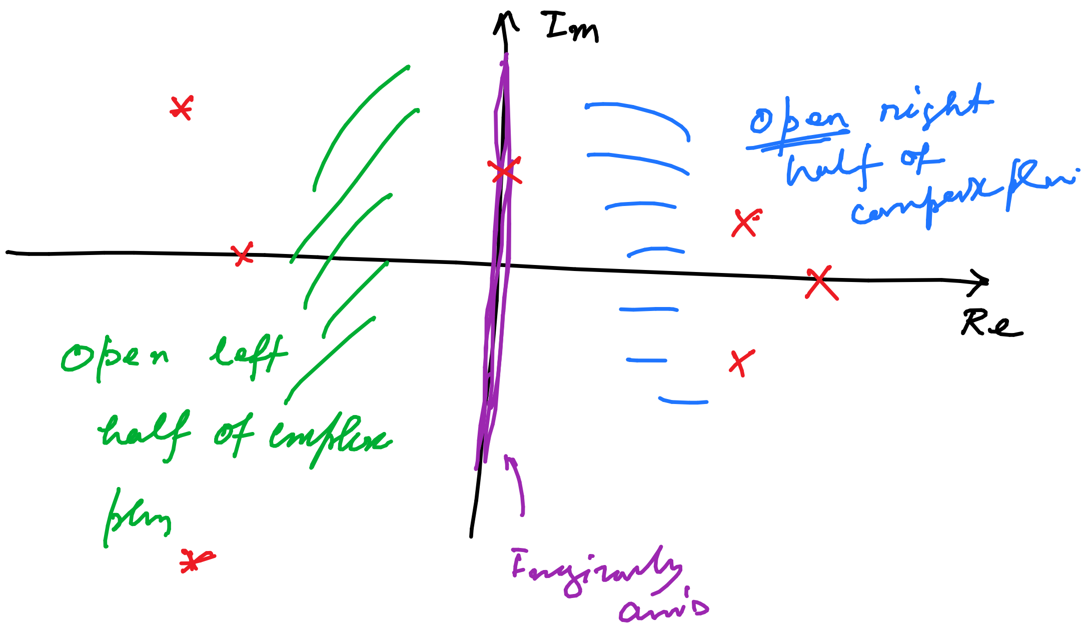
First column	$\epsilon = +ve$	$\epsilon = -ve$
1	$+$	$+$
2	$+$	$+$
ϵ	$+$	$-$
$\frac{6\epsilon - 7}{\epsilon}$	$-$	$+$
$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
3	$+$	$+$

Two sign changes in both cases.

The quantity ϵ could be +ve or -ve

In both cases the conclusion will be same.

(The number of sign changes will remain same)



s^* is a root of the polynomial $\alpha(s)$.

$\frac{1}{s^*}$ Does the sign of root change?

No — sign of real part of the root

$\rightarrow$ Let $\alpha(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

$s \rightarrow \frac{1}{\lambda}$ $\quad \alpha(\lambda) = \left(\frac{1}{\lambda}\right)^n + a_{n-1}\left(\frac{1}{\lambda}\right)^{n-1} + \cdots + a_1\left(\frac{1}{\lambda}\right) + a_0$

$$\alpha(\lambda) = \left(\frac{1}{\lambda}\right)^{n}\left[1 + a_{n-1}\left(\frac{1}{\lambda}\right)^{-1} + \cdots + a_1\left(\frac{1}{\lambda}\right)^{1-n} + a_0\left(\frac{1}{\lambda}\right)^{-n}\right]$$

$$= \left(\frac{1}{\lambda}\right)^{n}\Big[\underbrace{1 + a_{n-1}\lambda + a_{n-2}\lambda^{2} + \cdots + a_1\lambda^{n-1} + a_0\lambda^{n}}_{\tilde{\alpha}(\lambda)}\Big]$$

If λ is root of $\alpha(s)$, then $\frac{1}{\lambda}$ is a root of $\tilde{\alpha}(\lambda)$.

- A polynomial ($\tilde{\alpha}(\lambda)$) that has reciprocal roots of the original polynomial ($\alpha(s)$) has its roots distributed in same right half, left half & imaginary axis.

→ The roots will not move from one region (open left half of $\mathbb{C}$, imaginary axis and open right half of $\mathbb{C}$) to another.

→ In this process, there is a possibility - one can avoid using ϵ.

$$\alpha(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

$$s \to \frac{1}{\lambda}$$

$$\tilde{\alpha}(\lambda) = 3\lambda^5 + 5\lambda^4 + 6\lambda^3 + 3\lambda^2 + 2\lambda + 1$$

$\downarrow$

Construct Routh table for $\tilde{\alpha}(\lambda)$

λ^5	3	6	2
λ^4	5	3	1
λ^3	4.2	1.4	
λ^2	1.33	1	
λ^1	−1.75	0	
λ^0	1		

Same conclusion that two sign changes, & hence two roots of $\tilde{\alpha}(\lambda)$ on $\alpha(s)$ are in open right half of $\mathbb{C}$.

Roots of $\tilde{\alpha}(\lambda)$: $-0.6769 \pm 0.9339i$

$0.1433 \pm 0.6304i$

-0.5995

$$-\frac{1}{0.5995} = -1.6681 \leftarrow \text{root of } \alpha(s)$$

$\downarrow$

$$\frac{1}{-0.6769 \pm 0.9339i} = -0.5088 \mp 0.7020i$$

$$p(s) = s^3 + 10s^2 + 31s + 1030$$

$$p_o(s) = s^3 + 31s \leftarrow \text{Odd polyn.}$$

$$p_e(s) = 10s^2 + 1030 \leftarrow \text{Even polynomia}$$

$$10s^2 + 1030 \overline{\smash{\big)}\, s^3 + 31s \,} \left(\tfrac{1}{10}s \right. \leftarrow \text{quotient } q_1(s)$$

$$\begin{array}{l} s^3 + 103s \\ \hline -72s \end{array} \leftarrow \text{Remainder } p_3(s)$$

polynomial division

3rd row is constructed by taking the coefficients of remainder polynomial $p_3(s)$.

in constructing Routh table