ELL 225

Lecture - 23

- Contour mapping from one complex plane to another complex plane through a mapping function $F(s)$.

$$\rightarrow \quad F(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$\downarrow$

- Cauchy's Principle of Argument:

$\downarrow$

$$N = Z - P$$

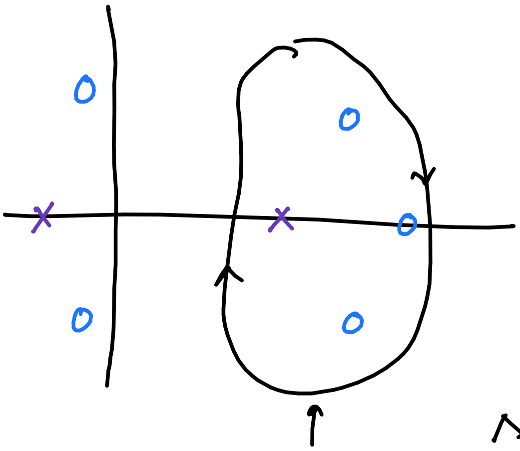
N: No. of encirclement of origin in <u>clockwise</u> direction by contour C_F in $F(s)$-plane.

Z: No. of zeros inside C_s in s-plane

P: No of poles inside C_s in s-plane

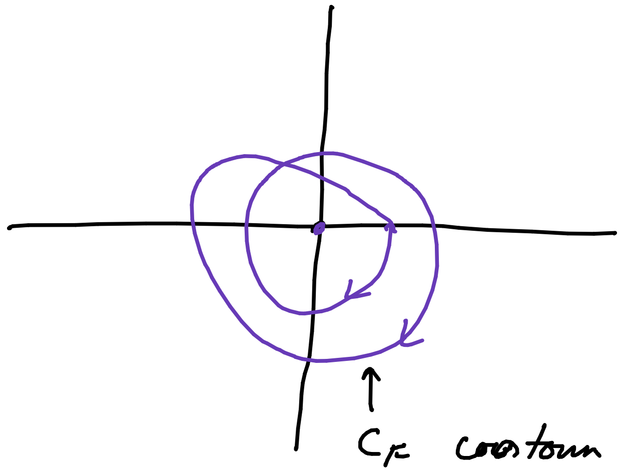
$\Downarrow$

One can get idea about the encirclement of origin by C_F contour in $F(s)$-plane in clockwise direction by looking at the number of poles & zeros inside the contour C_s in s-plane.



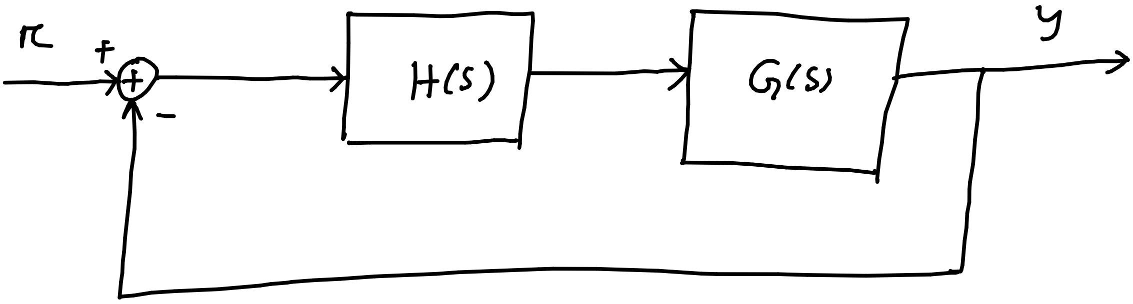
C_s - contour in s-plane

$$N = Z - P = 3 - 1 = 2$$



C_F contour in F(s)-plane

→ Use Principle of Argument in studying the stability of following closed loop system (BIBO stability)



- Assume that the poles & zeros of G(s) and H(s) are given in factored form (know the exact location of poles & zeros)

- We will investigate the poles of closed loop system, which belong to the closed right half of complex plane.

Let $G(s) = \dfrac{n_g(s)}{d_g(s)}$ $\quad H(s) = \dfrac{n_h(s)}{d_h(s)}$

The closed loop transfer funct^n

$$T(s) = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{n_g n_h}{d_g d_h + n_g n_h}$$

- The closed loop poles are the roots of the polynomial $\sigma(s) := d_g d_h + n_g n_h$

  the roots of $\sigma(s)$ are not in factored form.

So it is not clear where the roots of $\sigma(s)$ are.

- To study stability, we need to investigate the locations of roots of $\sigma(s)$ or closed loop poles.

$\downarrow$

To use Principle of Argument we need to define an appropriate mapping function.

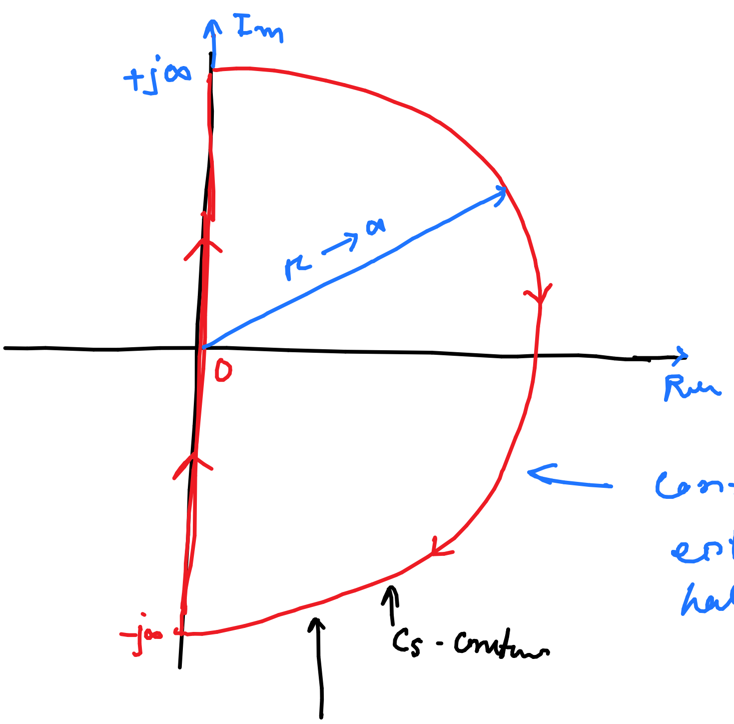
$$1 + G(s)H(s) = 1 + \frac{n_g n_h}{d_g d_h} = \frac{\overbrace{d_g d_h + n_g n_h}^{\sigma(s)\ \leftarrow\ \text{unknown}}}{\underbrace{d_g d_h}_{\leftarrow\ \text{known}}}$$

- The zeros of $1 + G(s)H(s) \equiv$ the closed loop poles ?
- The poles of $1 + G(s)H(s) \equiv$ the poles of $G(s)H(s)$ — known

* Let us define our mapping func'

as $\boxed{F(s) := 1 + G(s)H(s)}$

- To ensure stability, none of the roots of $\sigma(s)$ or the zeros of $F(s)$ should belong to the closed right half of complex plane.



radius : r

contour covering the entire closed right half of complex plane.

s-plane Nyquist contour or Nyquist path

$Z :=$ Number of zeros of $F(s) = 1 + G(s)H(s)$, which are inside the Nyquist contour
(= closed loop poles)

$P :=$ Number of poles of $F(s) = 1 + G(s)H(s)$, which are inside the Nyquist contour.

- Assume that the Nyquist contour C_s is mapped to a contour C_F in $F(s)$-plane

Nyquist plot.
Nyquist diagram

→ According to the principle of argument, the number of closed loop poles (Z) inside the Nyquist contour is equal to the number of enclosed poles (P) of $F(s)$ inside Nyquist contour + number of clock wise rotation of C_F - contour in $F(s)$-plane about the origin.

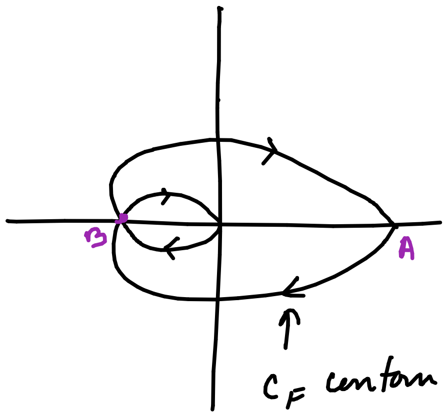
$F(s) = 1 + GH$

$$\boxed{Z = P + N}$$

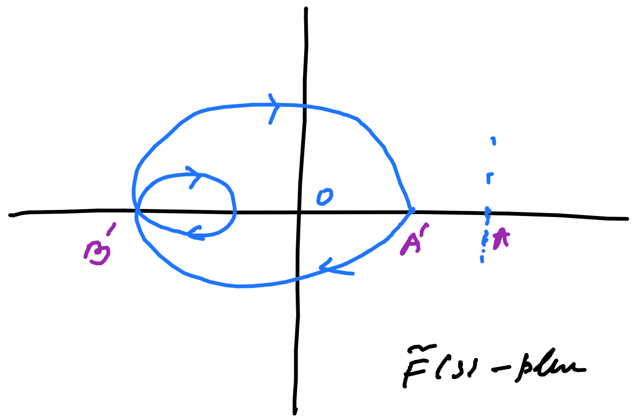
Equal to the zeros of $F(s)$, which needs to be determined
||
closed loop poles

equal to the poles of $F(s)$, which are known apriori

Needs to be evaluated by mapping the Nyquist contour to $F(s)$-plane.



$F(s)$ - plane



1 unit shifted towards left

$F(s) = 1 + G(s)H(s)$

$\tilde{F}(s) = G(s)H(s) = F(s) - 1$

Let $F(s) = x + jy \longrightarrow \tilde{F}(s) = \underline{(x-1)} + jy = v = M\angle\theta$

* Let us now define a new mapping function: $\boxed{\tilde{F}(s) = G(s)H(s)}$

Then the Nyquist plot C_F will get shifted 1 unit towards left in the $\tilde{F}(s)$ plane.

$\Downarrow$

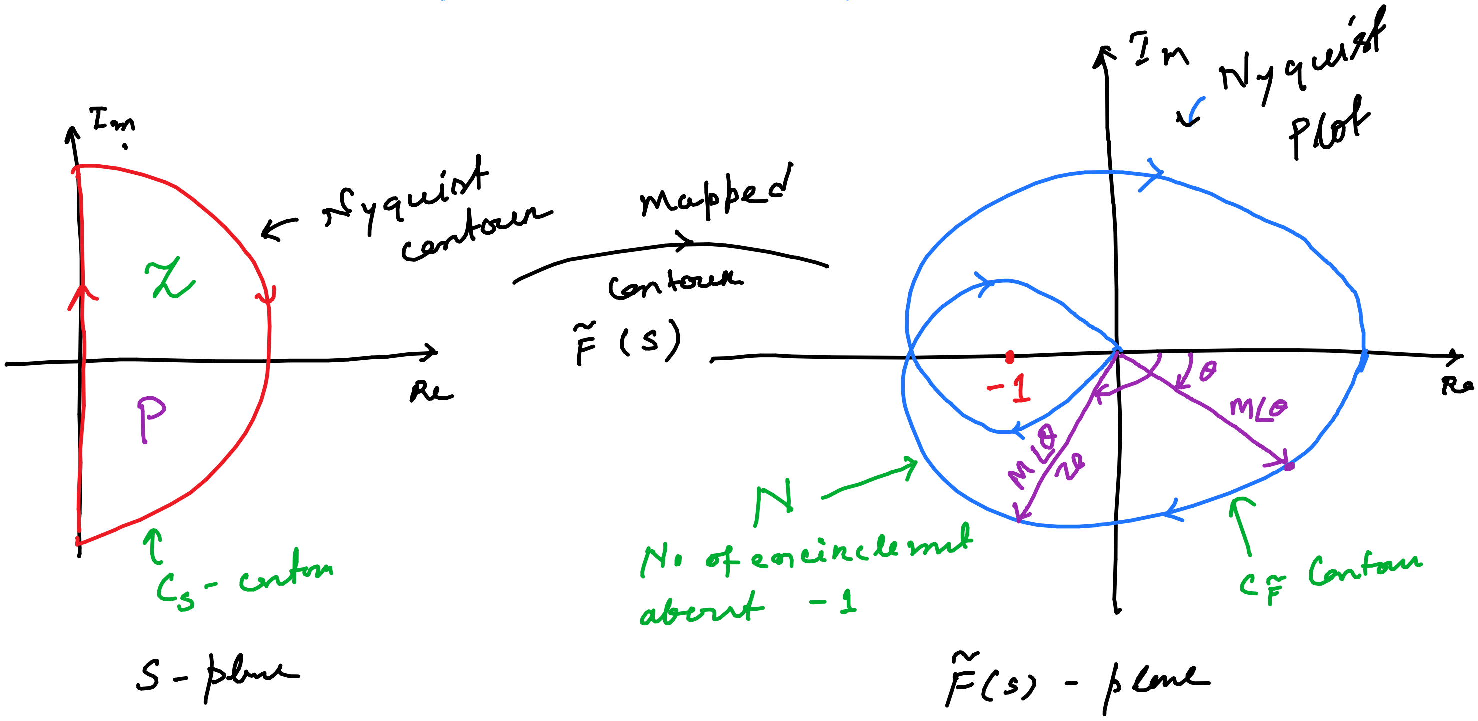
We need to look at the encirclement about the point $(-1 \pm j0)$ instead of origin.

Notation: closed right half of complex plane (including imaginary axis) $\rightarrow \mathbb{C}^+$

→ Nyquist Criterion (by H. Nyquist, 1932)

If the Nyquist contour (that encloses $\mathbb{C}^+$) C_s is mapped through a function $\tilde{F}(s) = G(s)H(s)$, then the number of closed loop poles (Z), in the $\mathbb{C}^+$ is equal to the number of poles of $\tilde{F}(s)$ (P), in $\mathbb{C}^+$ plus the number of <u>clockwise</u> rotation (N) of Nyquist plot $C_{\tilde{F}}$ (in $\tilde{F}(s)$-plane) about the point $(-1 \pm j0)$. i.e.

$$\boxed{Z = P + N}$$



→ Applying the Nyquist criterion to Determine Stability

To make the system <u>stable</u> we need

$$\boxed{Z = 0 .}$$

- if $P = 0$, then $N = 0$
- if P poles then $N = -P$

(1) The closed loop system is stable if and only if the contour $C_{\tilde{F}}$ (Nyquist plot) <u>does not</u> encircle the point $-1 \pm j0$, when the number of <u>poles of $G(s)H(s)$</u> in the $\mathbb{C}^+$ is zero, i.e. $P = 0$.
(No poles of $\tilde{F}(s)$ are inside Nyquist contour)

(2) The closed loop system is stable (BIBO) if and only if for the contour $C_{\tilde{F}}$, the number of <u>Counter-clockwise</u> encirclement of the point $(-1 \pm j0)$ is equal to the number of poles of $\tilde{F}(s) = G(s)H(s)$ with positive real parts. (poles of $F(s)$ are inside the Nyquist contour)
i.e. <u>$N = -P$</u>.