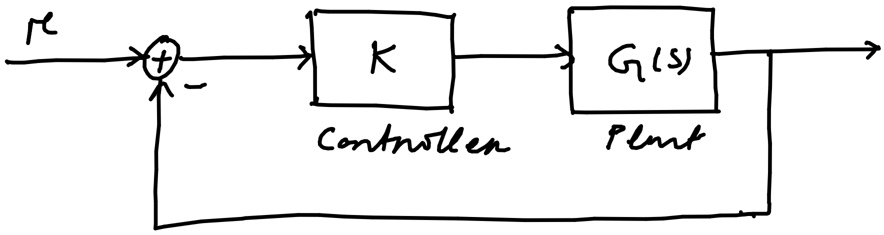
ELL225

Lecture - 25

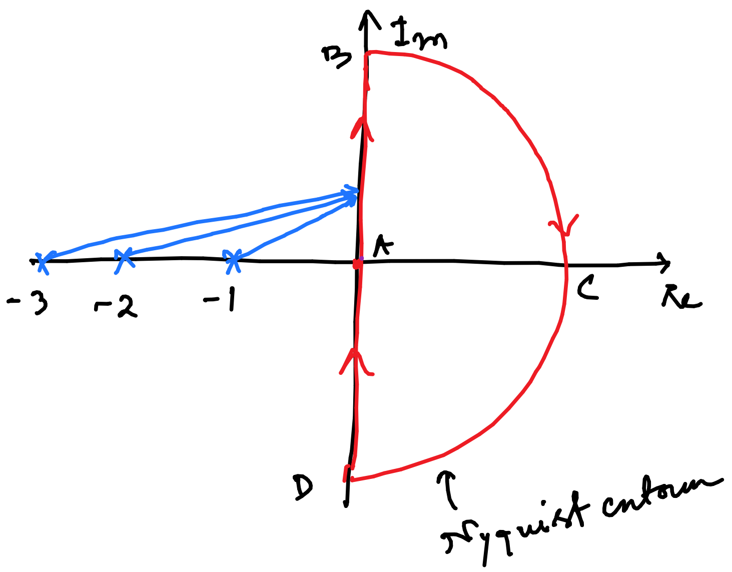
Nyquist contour mapping

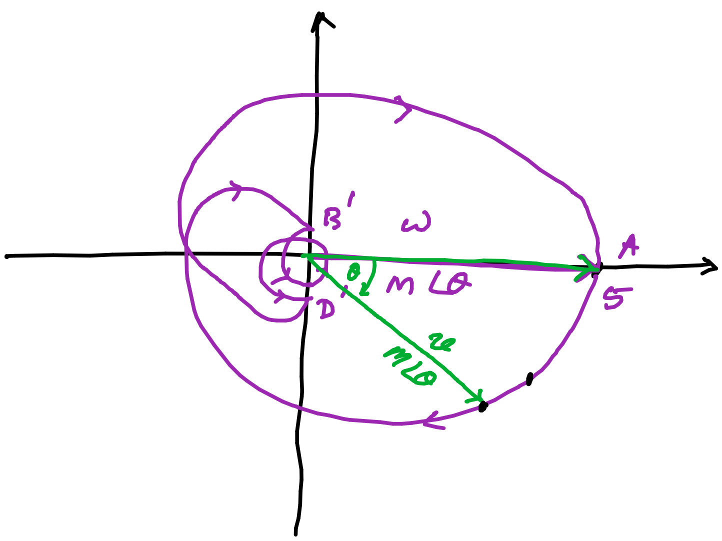


$v = \tilde{F}(s) = K\,G(s)$ $\qquad\qquad v = x + jy = M\angle\theta$

Let $\qquad G(s) = \dfrac{30}{(s+1)(s+2)(s+3)}$

Assume that $K = 1$





At point A

$M = \dfrac{30}{1\times2\times3} = 5 \qquad \theta = 0^\circ$

For $A \to B$

$\theta = -\theta_{p_1} - \theta_{p_2} - \theta_{p_3}$

$= -270^\circ$

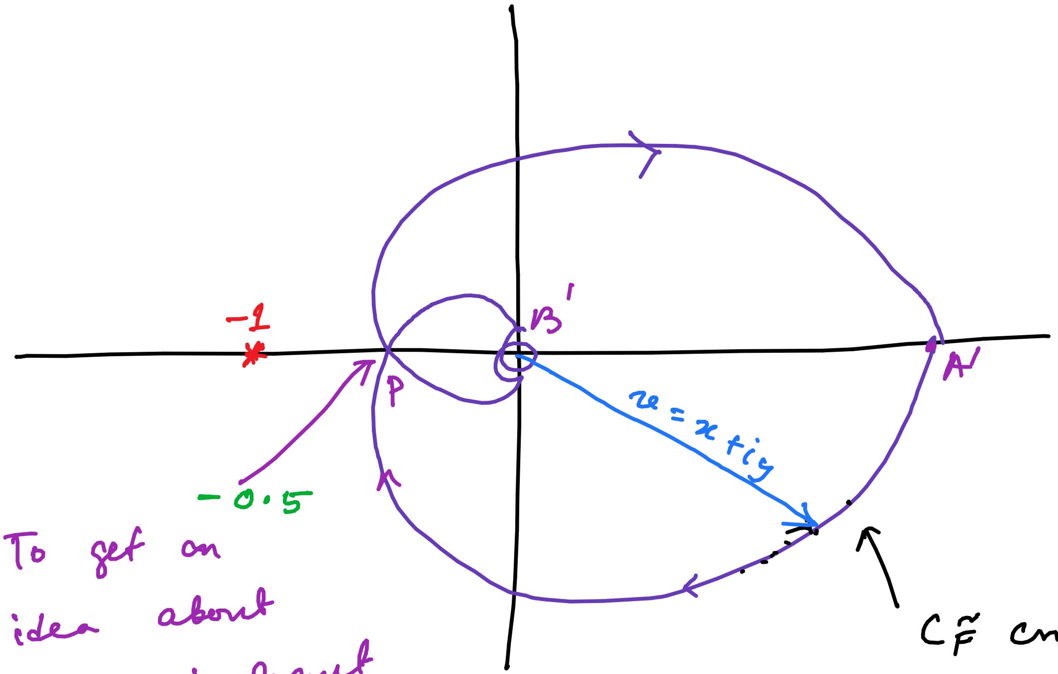
For B - D

M is very small $\theta = -(-180^\circ) - (-180^\circ) - (-180^\circ)$

$= 3 \times 180$

$= 540$

→ When we study stability the mapped Nyquist contour from B to D (having ∞ radius) does not contribute for encirclement about point -1. (since its magnitude is very small & the contour remains very close to the origin)



To get an idea about the encirclement of point -1, we need to compute the point P.

CF contour of Nyquist plot

$$G(j\omega) = \frac{30}{(1+j\omega)(2+j\omega)(3+j\omega)}$$

why s is replaced with only $j\omega$ instead of $\sigma + j\omega$?

Since we are moving from A to B, $\sigma = 0$

To evaluate point P, where $v = x+jy$ has 0 imaginary part, we will make the imaginary part of $G(j\omega)$ to 0.

$$v = G(j\omega) = 30\left[\frac{(1-j\omega)(2-j\omega)(3-j\omega)}{(1+j\omega)(1-j\omega)(2+j\omega)(2-j\omega)(3+j\omega)(3-j\omega)}\right]$$

$$= 30\left[\frac{6-6\omega^2+j\omega^3-j11\omega}{(1+\omega^2)(4+\omega^2)(9+\omega^2)}\right]$$

$$w = G(j\omega) = 30\left[\underbrace{\frac{6-6\omega^2}{(1+\omega^2)(4+\omega^2)(9+\omega^2)}}_{x} + j\underbrace{\frac{\omega^3-11\omega}{(1+\omega^2)(4+\omega^2)(9+\omega^2)}}_{y}\right]$$

Set $y = 0$

then $\omega^3 - 11\omega = 0$

$\Rightarrow \omega^2 = 11 \qquad \omega = \sqrt{11}$

At $\omega = \sqrt{11}$ the vector $w = x + jy$ has reached at point P.

- Then the real part of $w = x+jy$ at point P

is $G(j\omega) = 30\left[\frac{6-66}{12 \times 15 \times 20}\right] = -0.5$

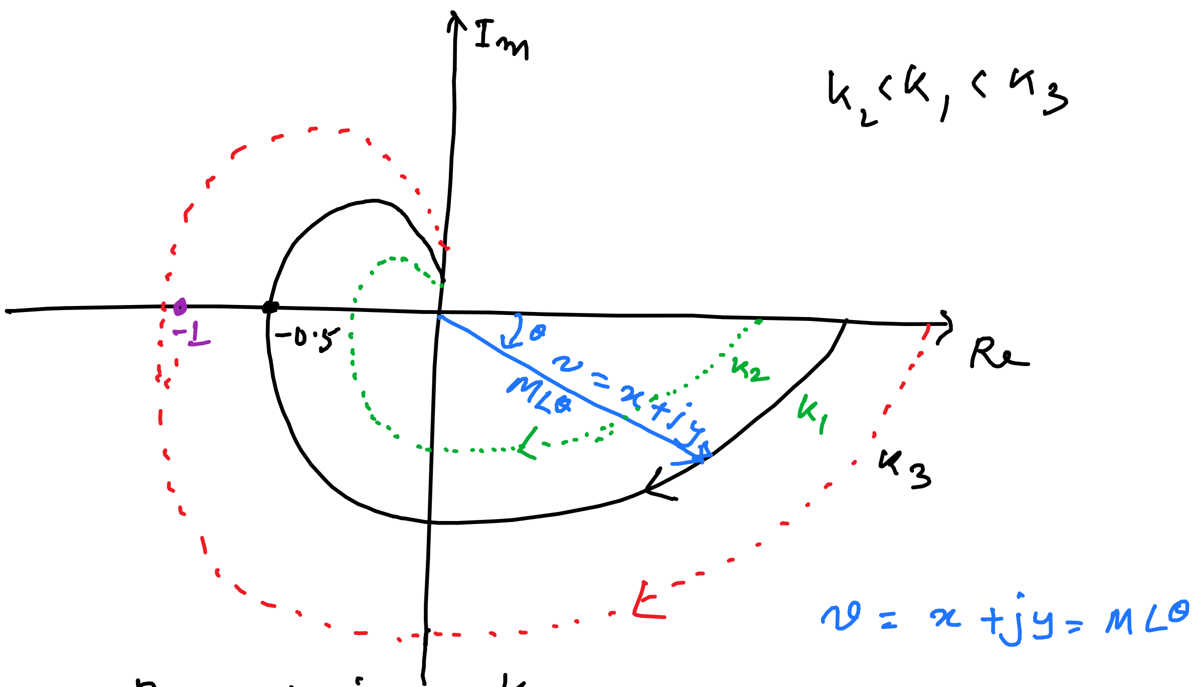
$\Downarrow$

There is no encirclement about the point $-1 \pm j0$ by Nyquist plot on $C_{\sim}$ contour.

$$Z = P + N$$
$$= 0 + 0$$

$\Rightarrow$ $Z = 0$ $\Rightarrow$ The closed loop system for $K = 1$ is stable, which follows from Nyquist stability criterion.

→ We will now vary the gain K.



By changing K the magnitude of v will change & the angle θ will remain same.

The closed loop system will remain stable as long as there is no encirclement of point $-1 \pm j0$ for changing K.

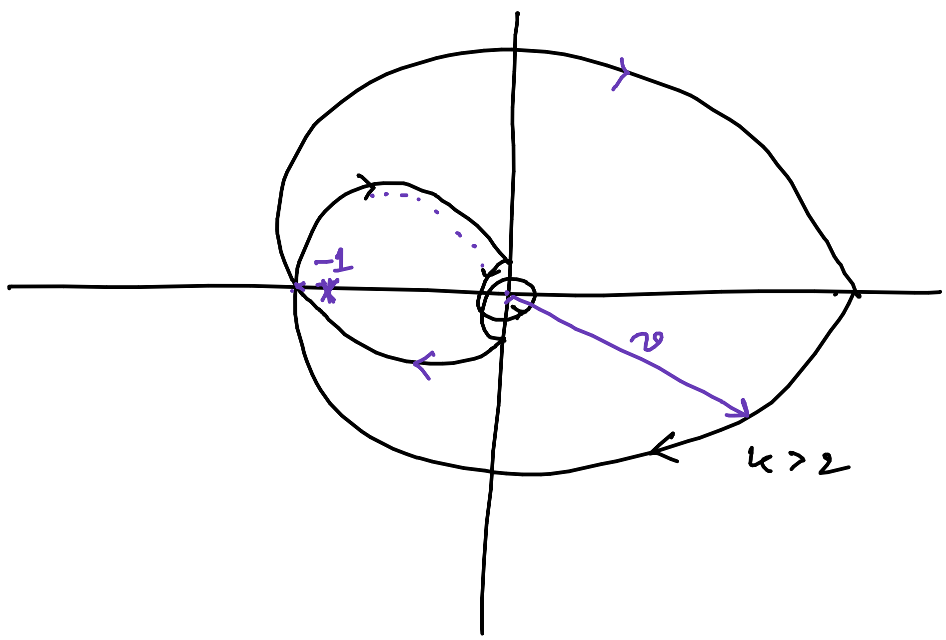
- For $0 < K \leq 2$ there will be no encirclement about point -1

BIBO stable

$K > 2$ there will be encirclement of point -1.

→ Point P for $k < 2$ will be greater than -1

point P for $k > 2$ will be less than -1



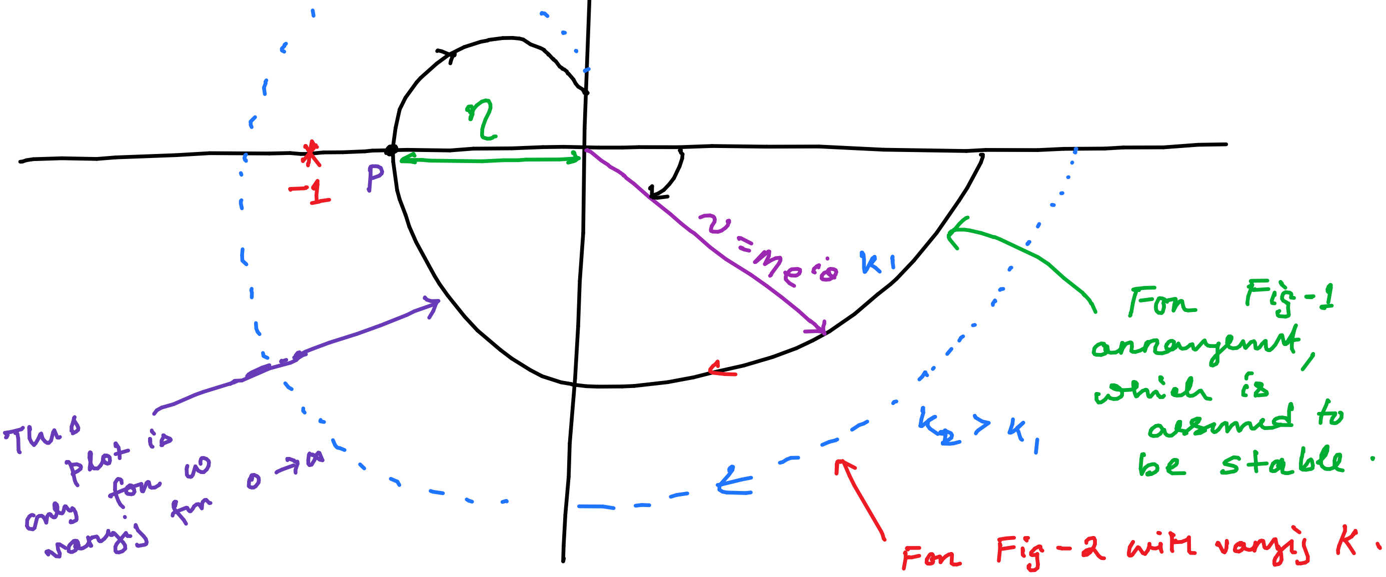
$N = 2$

$Z = P + N$
$= 0 + 2$
$= 2$ which is unstable.

→ Gain Margin (G.M.)

$v = \tilde{F}(j\omega) = M \angle \theta$

$M = |\tilde{F}(j\omega)| = \eta$



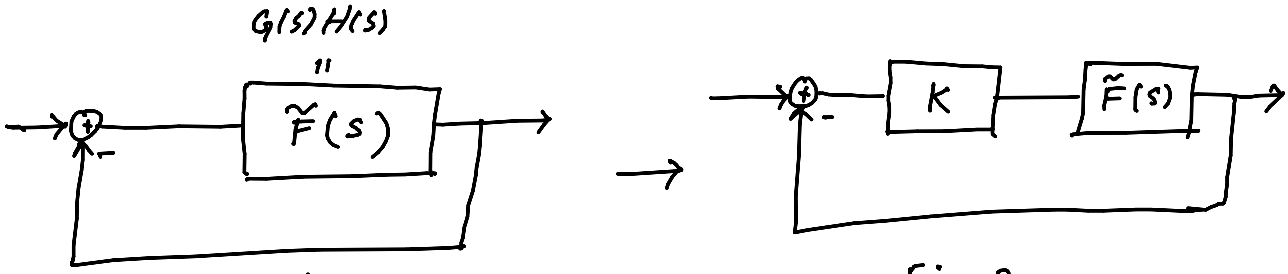


Fig - 1 Fig - 2

- By changing the parameter $K > 0$, one can make the Nyquist plot to encircle point $-1, 0$ hence change the stability property (stable → unstable).

→ At point P, we have $|v| = \underbrace{|\tilde{F}(j\omega)|}_{\text{gain of } \tilde{F}(j\omega) = G(j\omega)H(j\omega)} = \eta$

$\Downarrow$

- If we change the gain of $\tilde{F}(j\omega)$ by an amount of $\frac{1}{|\tilde{F}(j\omega)|} = \frac{1}{\eta}$, then system will be in the verge of instability.

- <u>Gain Margin</u> : It is the amount of gain change required of $\tilde{F}(j\omega) = G(j\omega)H(j\omega)$ at the phase angle $\theta = -180^\circ$ such that the closed loop system become (BIBO) unstable (the Nyquist plot will pass through the point $-1 \pm j0$).

$\theta = \angle \tilde{F}(j\omega)$

The point where Nyquist plot cross -ve real axis.

$\Updownarrow$

The reciprocal of gain $|\tilde{F}(j\omega)|$ at the frequency (denoted as ω_π) at which the phase angle $\theta = -180^\circ$.

$\Updownarrow$

$$\text{Gain margin} = \frac{1}{|\tilde{F}(j\omega_\pi)|}$$

↓ usually defined in terms of decibel

Hence Gain margin $= 20 \log \left(\frac{1}{\eta}\right)$

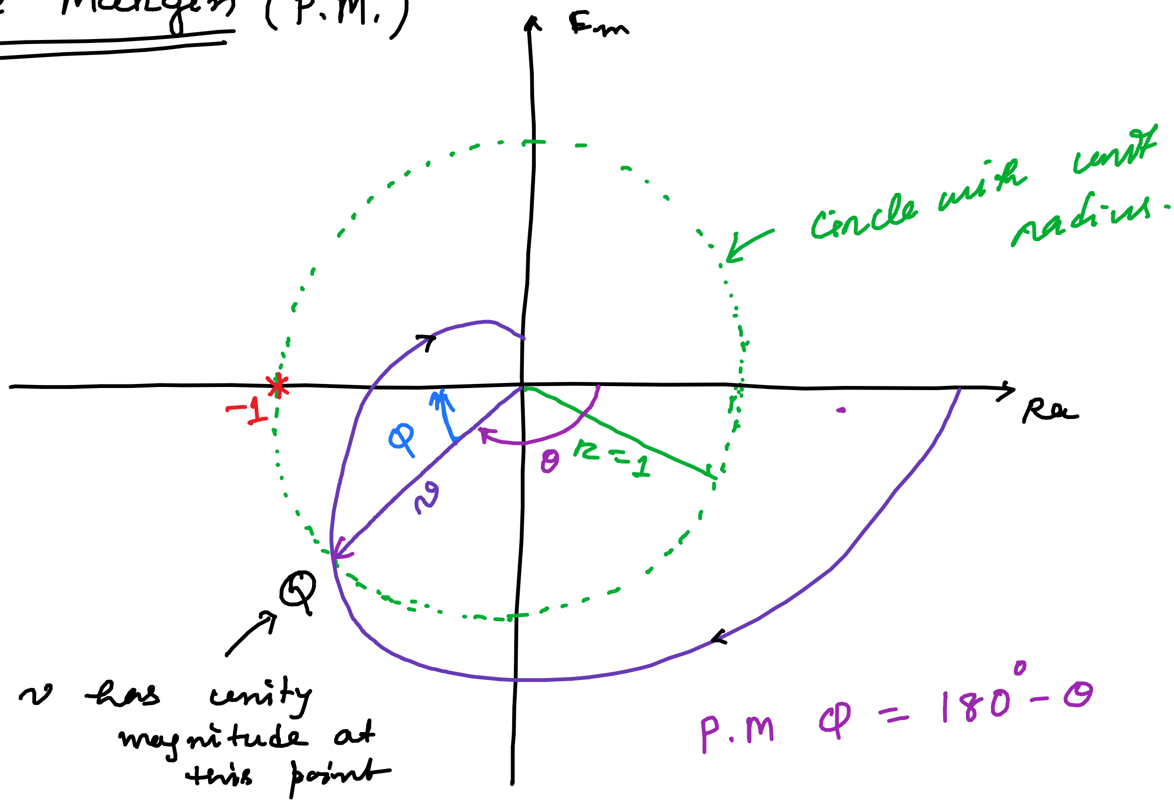
$= -20 \log \eta$ dB

where $\eta = |\tilde{F}(j\omega_\pi)|$

- ω_π : Phase cross-over frequency.

- Depending upon the value of η, the G.M. could be +ve or -ve (see the "log funt" property).

- **Phase Margin (P.M.)**



One can make the system unstable by changing the phase angle θ of $\tilde{F}(j\omega)$ at unity gain.

↓

For instance, in the above figure, if we hold the point Q & rotate it by an angle φ in clockwise direct? then the system will be in the range of instability, i.e. the Nyquist plot will pass through the point -1. Hence phase margin is defined as follows:

- Phase Margin : It is the amount of phase
(P.M.(φ)) shift of $\tilde{F}(j\omega)$ is required at unity gain ($|\tilde{F}(j\omega)| = 1$) to make the closed loop system unstable (or the Nyquist plot pass through the point $-1 \pm j0$.

or

The amount of angle that the Nyquist plot of $\tilde{F}(j\omega)$ must be rotated so that the unity magnitude point ($|\tilde{F}(j\omega)| = 1$, Q-point) will pass through the point $-1 \pm j0$.

* The G.M. & P.M. gives quantitative measure of closed loop system stability (measures for distance to closed loop instability).

→ Let ω_u be the frequency at which

$$|\tilde{F}(j\omega_u)| = 1 \quad \text{(Q-point)}$$

ω_u : Gain cross-over frequency.

In the previous fig :

$$-\theta - \varphi = -180^\circ$$

$$\boxed{\Rightarrow \varphi = 180^\circ - \theta}$$ ← Phase margin

Example :

$$G(s) = \frac{5}{s(s+1)(s+5)}$$

$$G(j\omega) = \frac{5}{(j\omega)(j\omega+1)(j\omega+5)}$$

$$= \frac{5(-j\omega)(1-j\omega)(5-j\omega)}{\omega^2(1+\omega^2)(25+\omega^2)}$$

$$-j5\omega(5 - j\omega - j5\omega - \omega^2)$$
$$= -j25\omega - 5\omega^2 - 25\omega^2 + j5\omega^3$$

$$= -\frac{30\omega^2}{\omega^2(1+\omega^2)(25+\omega^2)} + j\frac{5\omega(\omega^2-5)}{\omega^2(1+\omega^2)(25+\omega^2)}$$

Negative Real axis crossing $\mathrm{Im}(\cdot) = 0$

$$\Rightarrow 5\omega(\omega^2 - 5) = 0$$

$$\omega = \pm\sqrt{5} \rightarrow \omega = \sqrt{5}$$

(8.25)

$$\mathrm{Re}(G(j\sqrt{5})) = -\frac{150}{5\times 6\times 30} = -\frac{1}{6}$$

So $\eta = 0.167$

$$G.M. = -20\lg\eta = -20\lg(0.167) = 15.55\,dB$$

- To compute P.M., we need to find ω for which $|G(j\omega)| = 1$

$$\Rightarrow \left[\frac{30\omega^2}{\omega^2(1+\omega^2)(25+\omega^2)}\right]^2 + \left[\frac{5\omega(\omega^2-5)}{\omega^2(1+\omega^2)(25+\omega^2)}\right]^2 = 1$$

Needs to solve for ω (Computer may help you)

It is computed that for $\omega = 0.78$ we have $|G(j\omega)| \cong 1$.

Here for $\omega = 0.78$, we need to compute $\angle G(j\omega)$

* [Following rules may be used for $\tan^{-1}(\cdot)$: For complex no. $\sigma + j\omega$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{\omega}{\sigma}\right), & \text{for } \sigma > 0,\ \omega > 0 \quad (1^{st}\ \&\ 4^{th}\ \text{quadrant}) \\ \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \pi, & \text{for } \omega \geq 0,\ \sigma < 0 \quad (2^{nd}\ \text{quadrant}) \\ \tan^{-1}\left(\frac{\omega}{\sigma}\right) - \pi, & \text{for } \omega < 0,\ \sigma < 0 \quad (3^{rd}\ \text{quadrant}) \\ \pi/2, & \text{for } \omega > 0,\ \sigma = 0 \\ -\pi/2, & \text{for } \omega < 0,\ \sigma = 0 \end{cases}$$

For our case :

evaluating $G(j\omega)$ at $\omega = 0.78$

$$G(j\omega) = -\frac{30\omega^2}{\omega^2(1+\omega^2)(25+\omega^2)} + j\,\frac{5\omega(\omega^2-5)}{\omega^2(1+\omega^2)(25+\omega^2)}$$

$$= -\frac{18.252}{25.059} + j\left(\frac{-17.12}{25.059}\right)$$

Hence,

$$\angle G(j\omega) = \tan^{-1}\left(\frac{17.12}{18.252}\right) - \pi$$
$$= 43.167 - 180$$
$$= -136.83$$

$\Downarrow$

$\angle G(j\omega)$ can also be calculated according to our standard procedure:

$$\angle G(j\omega) = -\theta_{p_1} - \theta_{p_2} - \theta_{p_3}$$
$$= -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

For $\omega = 0.78$

$$\angle G(j\omega) = -90^\circ - 37.95^\circ - 8.86^\circ$$
$$= -136.81^\circ$$

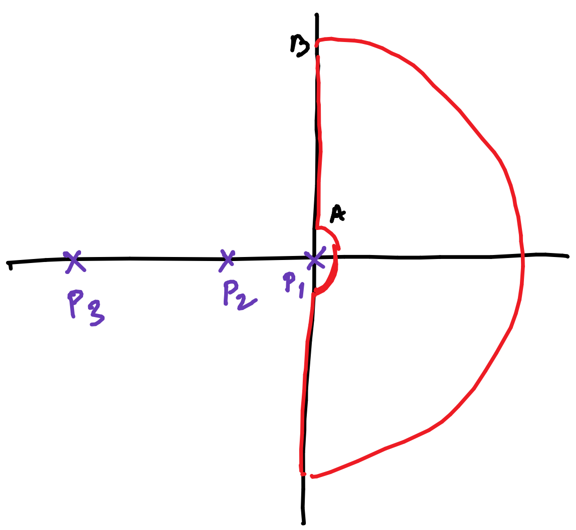
To compute P.M.:

$$\boxed{-136.83 - \phi = -180^\circ}$$
$$\Rightarrow \phi = 43.17^\circ$$

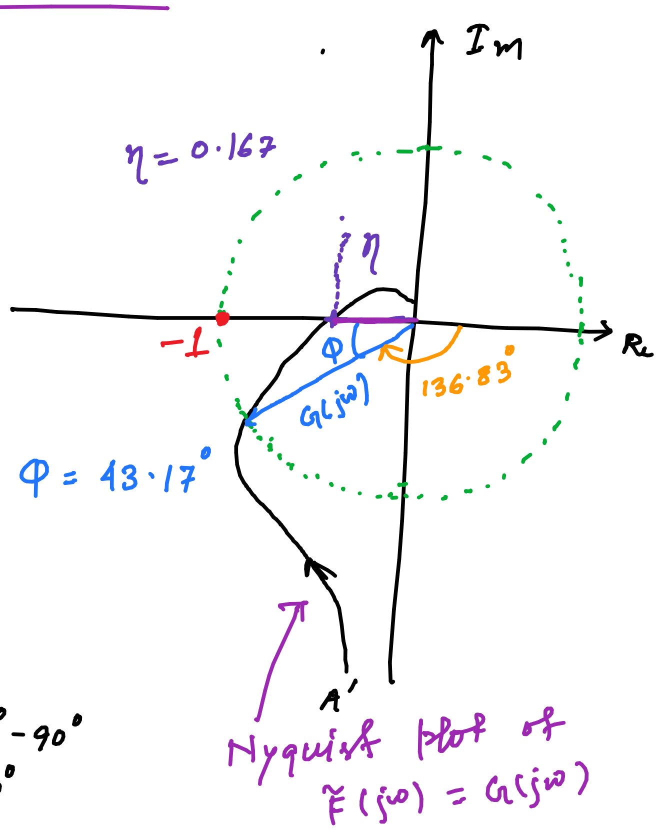
$\Downarrow$

43.17° additional phase shift required for $\tilde{F}(j\omega) = G(j\omega)$, so that the Nyquist plot of $\tilde{F}(j\omega)$ will pass through $-1 \pm j0$ & make the closed loop system (B/30) unstable.

$\Rightarrow$ Phase Margin $\phi = 43.17^\circ$



$\longrightarrow$

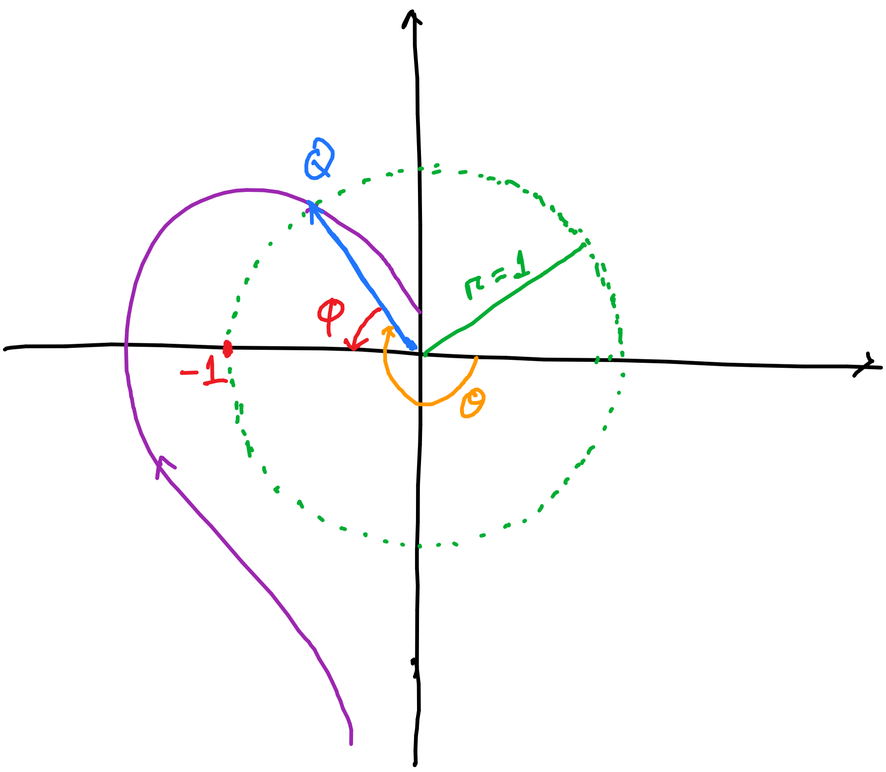


Point A $\rightarrow$ $\theta = -90^\circ$

$M \approx \infty$

For A$\rightarrow$B

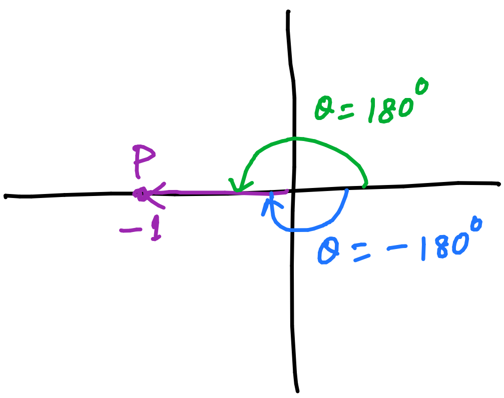
$$\theta = -90^\circ - 90^\circ$$
$$= -180^\circ$$



The point Q needs to be rotated an amount of $\varphi°$ in counter-clockwise direction to make the Nyquist plot pass through point -1.

$$-\theta + \varphi = -180°$$

$$\boxed{\Rightarrow \varphi = \theta - 180°}$$ Phase margin.



The point P has magnitude 1 & angle $\theta = \pm 180°$, depending upon the considered direction of rotation of vector $-1+j0$.

Here for defining G.M & P.M. the crucial phase angle -180^0 that we are considering, can also be considered as 180^0.

For instance

$$-\theta + \phi = 180^0 \quad \text{(in 2d fig for pm)}$$

P.M. $\boxed{\Rightarrow \phi = 180^0 + \theta}$