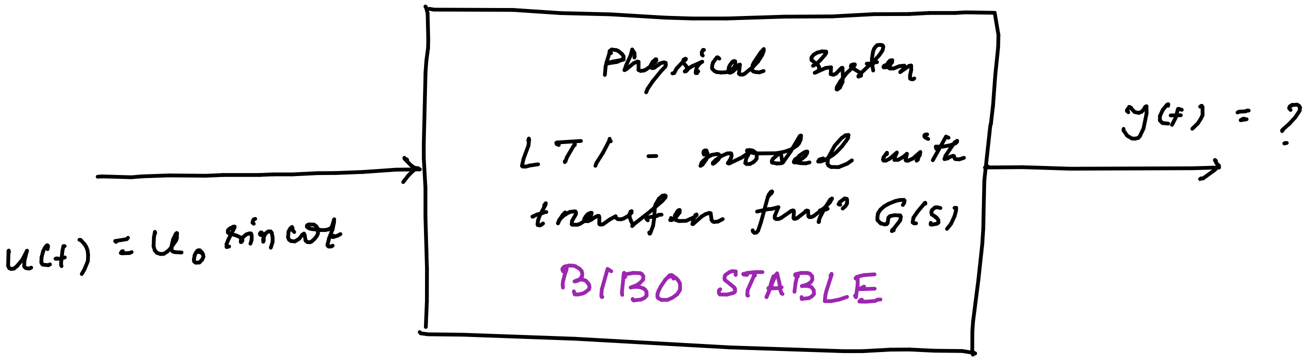
ELL 225

Lecture - 26



Let us assume that the transfer funct $G(s) = \dfrac{n(s)}{d(s)}$ has $\underbrace{\text{distinct poles}}_{\text{poles : } p_i}$.

$(\deg n(s) < \deg d(s))$

The output

$$Y(s) = G(s)\, U(s)$$

$$= G(s) \frac{u_0 \omega}{s^2 + \omega^2}$$

Using partial fraction expansion :

$$Y(s) = G(s) \underbrace{\frac{u_0 \omega}{s^2+\omega^2}}$$

$$= \underbrace{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n}$$

$\bar{a}$ is complex conjugate of a

We have assumed that $G(s)$ is BIBO stable

$\Downarrow$

the poles p_i are in open left half of $\mathbb{C}$.

$$Y(s) = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \cdots + \frac{b_n}{s+p_n}$$

$\mathcal{L}^{-1} \downarrow$

$$y(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t} + \underbrace{b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \cdots + b_n e^{-p_n t}}$$

At steady state ($t \to \infty$) these terms will decay down to 0, since p_i have -ve real parts.

At steady state:

$$\boxed{y_{ss}(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}}$$

To find a & $\bar{a}$, let us take the Laplace transform:

$$Y_{ss}(s) = \underbrace{G(s) \frac{\omega u_0}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}}$$

$$\Rightarrow G(s)\, \omega u_0 = a(s-j\omega) + \bar{a}(s+j\omega) \quad \cdots (*)$$

For $s = -j\omega$

$$G(-j\omega)\,\omega u_0 = a\,(-j\omega - j\omega)$$

$$\boxed{\Rightarrow \quad a = -\frac{G(-j\omega)\,u_0}{2j}}$$

For $s = j\omega$

$$G(j\omega)\,\omega u_0 = \bar{a}\,(2j\omega)$$

$$\boxed{\Rightarrow \quad \bar{a} = \frac{G(j\omega)\,u_0}{2j}}$$

For any $\omega > 0$, $G(j\omega)$ is a complex number, which can be represented as

$$G(j\omega) \rightarrow |G(j\omega)|\, e^{j\varphi}$$

where $\varphi = \angle G(j\omega) = \tan^{-1}\left[\frac{\text{Im. part of } G(j\omega)}{\text{Real part of } G(j\omega)}\right]$

$$G(-j\omega) = |G(j\omega)|\, e^{-j\varphi}$$

Then

$$a = -\frac{u_0 |G(j\omega)| e^{-j\varphi}}{2j}$$

$$\bar{a} = \frac{u_0 |G(j\omega)| e^{j\varphi}}{2j}$$

$$y_{ss}(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}$$

$$= -\frac{u_0 |G(j\omega)| e^{-j\varphi}}{2j} e^{-j\omega t} + \frac{u_0 |G(j\omega)| e^{j\varphi}}{2j} e^{j\omega t}$$

$$= u_0 |G(j\omega)| \frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j}$$

$$= \underbrace{u_0 |G(j\omega)|}_{y_0} \sin(\omega t + \varphi)$$

$$= y_0 \sin(\omega t + \varphi)$$

- At steady state, the output response of an LTI BIBO stable system will produce

$$y_{ss}(t) = y_0 \sin(\omega t + \varphi) \quad \text{for an input}$$

$$u(t) = u_0 \sin \omega t$$

when $\boxed{y_0 = u_0 |G(j\omega)| \qquad \varphi = \angle G(j\omega)}$

We will see only changes in magnitude y_0 and phase angle φ.

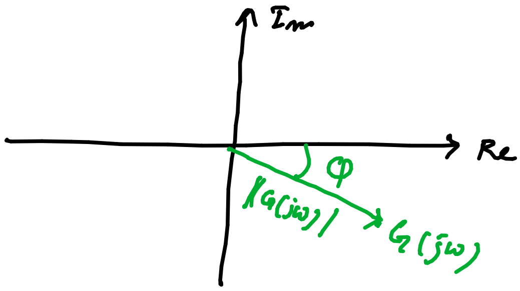
- Both $y_0 = u_0 |G(j\omega)|$ and $\varphi = \angle G(j\omega)$ are funct^n of frequency.

- By changing the frequency, we will only see changes in $|G(j\omega)|$ & $\angle G(j\omega)$

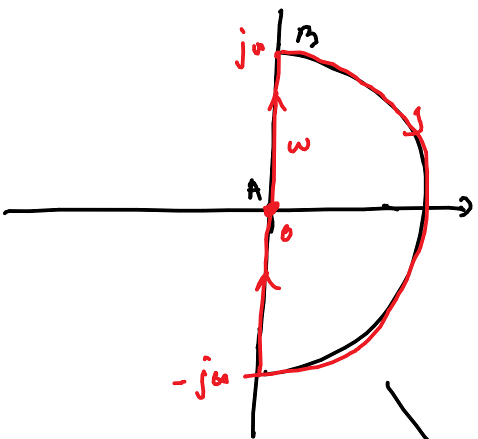
$\downarrow$

To plot $|G(j\omega)|$ & $\angle G(j\omega)$ we can

- plot in complex plane

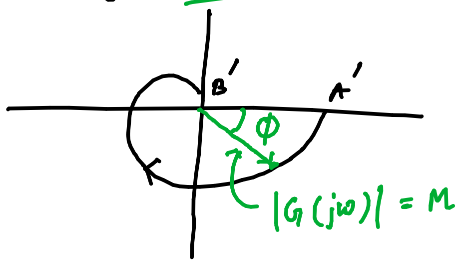


← we studied in Nyquist plot



we are only interested for frequency range $\underline{\underline{0 \leq \omega < \infty}}$

mapped through $\underline{\underline{G(j\omega)}}$



- The magnitude & phase angles of $G(j\omega)$ are combined together & represented in complex plane. ← Polar plot

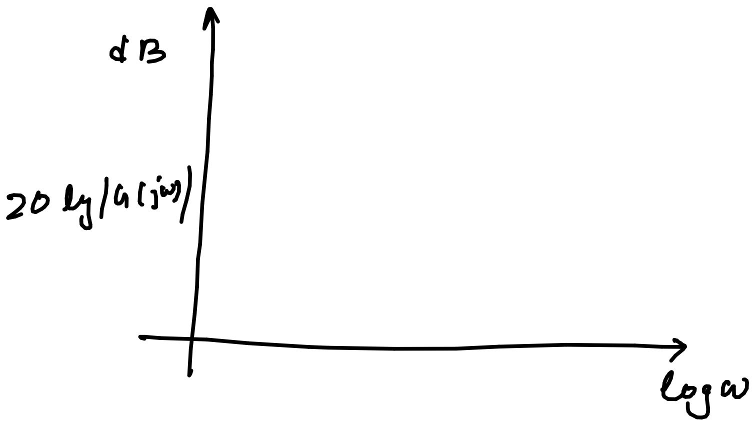
- Also the magnitude & phase can be plotted separately for varying frequency.

Usually the magnitudes are expressed in terms of dB.

↳ $|G(j\omega)| \rightarrow 20 \log |G(j\omega)|$ dB (decibel)

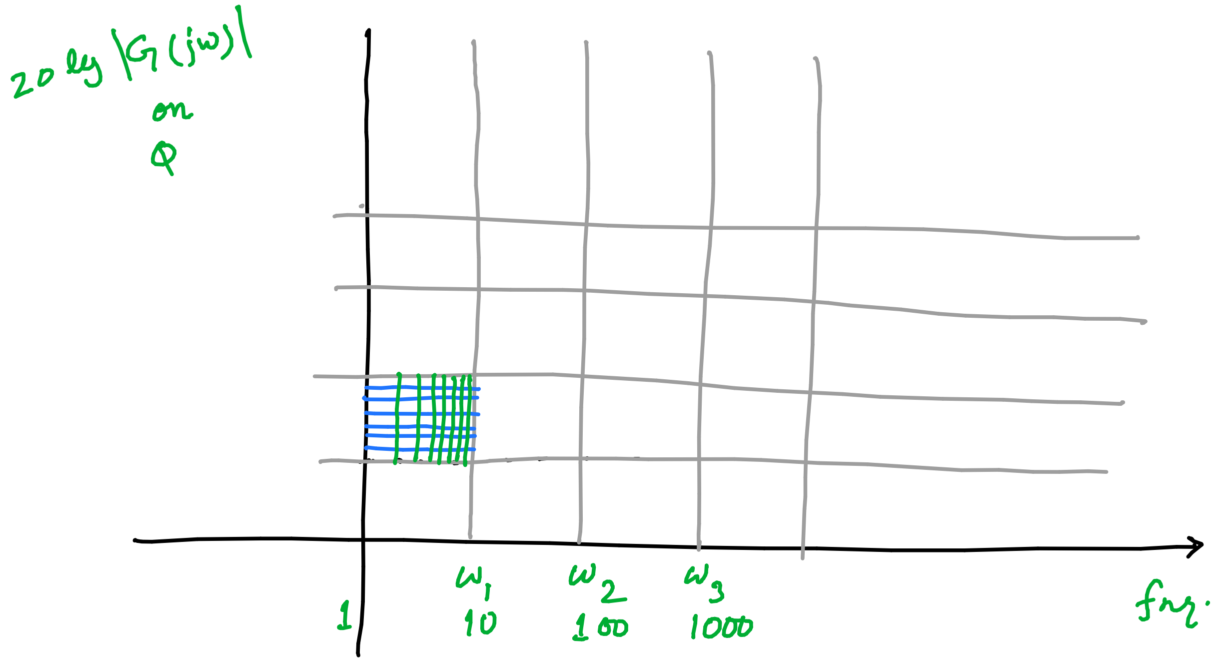
↓

- The magnitude & phase of $G(j\omega)$ will be plotted in terms of dB ~ $\log \omega$.



↓

We use semi-log paper, where the horizontal axis is in terms of logarithmic scale & vertical axis is in linear scale.



$$\log \omega_2 - \log \omega_1$$

$$= 2\log 10 - \log 10 = 1$$

"decade"

$$\log \omega_3 - \log \omega_2$$

$$= 3\log 10 - 2\log 10$$

$$= 1$$

"decade"

→ Plotting of frequency response (by varying ω) plot:

Bode plot

- magnitude plot $\quad 20\log|G(j\omega)| \sim \log\omega$
- phase plot $\quad \phi = \angle G(j\omega) \sim \log\omega$

Assume that

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{s^N(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$|G(j\omega)| = \left.\frac{|K||(s+z_1)||(s+z_2)|\cdots|(s+z_m)|}{|s^N||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|}\right|_{s\to j\omega}$$

$$\log(A/B) = \log A - \log B$$

$$\log(AB) = \log A + \log B$$

$$20\log|G(j\omega)| = 20\log|K| + 20\log|j\omega + z_1| + \cdots + 20\log|j\omega + z_m|$$
$$- 20\log|(j\omega)^N| - 20\log|j\omega + p_1| - \cdots - 20\log|j\omega + p_n|$$

$$\varphi = \sum_{i=1}^{m} \theta_{z_i} - \sum_{k}^{n+N} \theta_{p_i}$$

→ Basic factors that appear in $G(s)$

- Gain K
- Integral & derivative factor $\frac{1}{s}$ or s $\left((j\omega)^{\pm 1}\right)$
- First order factors $\underbrace{|s + z_i| \text{ or } \frac{1}{|s+p_i|}}_{(1 + j\omega\tau)^{\pm 1}}$
- Quadrati terms $\left(1 + 2\xi(j\omega/\omega_n) + j(\omega/\omega_n)^2\right)^{\pm 1}$

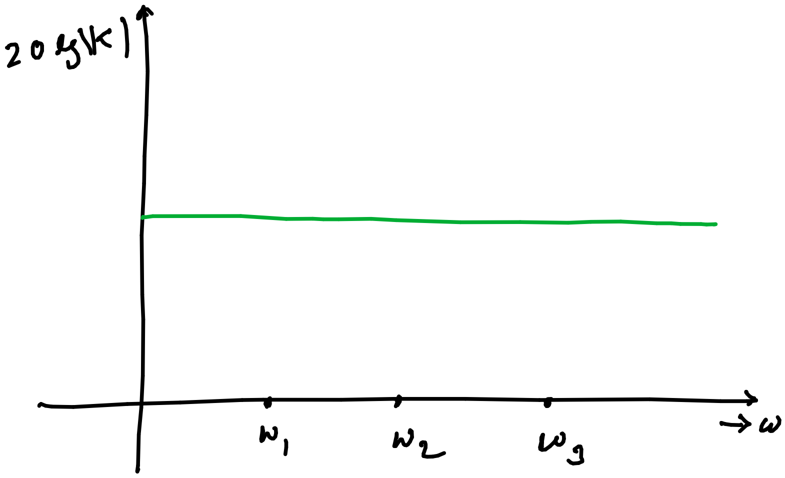
→ For gain K

Evaluate $20\log|K|$ → at different frequencies

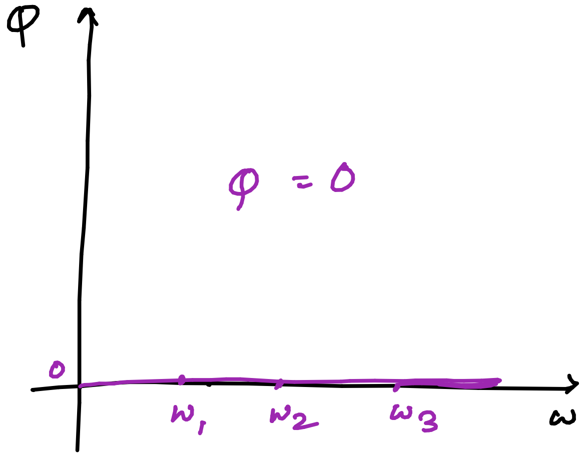
$\omega_1 = 0.1$

$\omega_2 = 10$

$\omega_3 = 100$



Magnitude plot



Phase plot