ELL225

Lecture - 31

→ Behavior of Root-locus at infinity:

$s \to \infty$

$$K\,G(s)H(s) = \frac{K}{s(s+1)(s+3)}$$

$$\approx \frac{K}{s \cdot s \cdot s} \quad \text{as} \quad s \to \infty$$

⇓

It has 3 zeros at infinity

It has 3 poles at finite locations.

Where are these zeros?

→ **ASYMPTOTES**

The branches of R.L. approach a set of straight lines (Asymptotes) as 's' go far away from origin ($s \to \infty$). These asymptotes start from a point on real axis, called center of asymptotes σ_a, which is calculated as:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\underbrace{\text{No. of finite poles}}_{n} - \underbrace{\text{No. of finite zeros}}_{m}}$$

• The angles between the asymptotes & real axis :

$$\theta_a = \frac{(2l+1)\pi}{n-m}, \quad l = 0, 1, \cdots, n-m-1$$

• Total $\underline{\underline{n-m}}$ asymptotes

$$K\,G(s)\,H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \qquad \theta_a = \frac{(2l+1)\pi}{3}$$

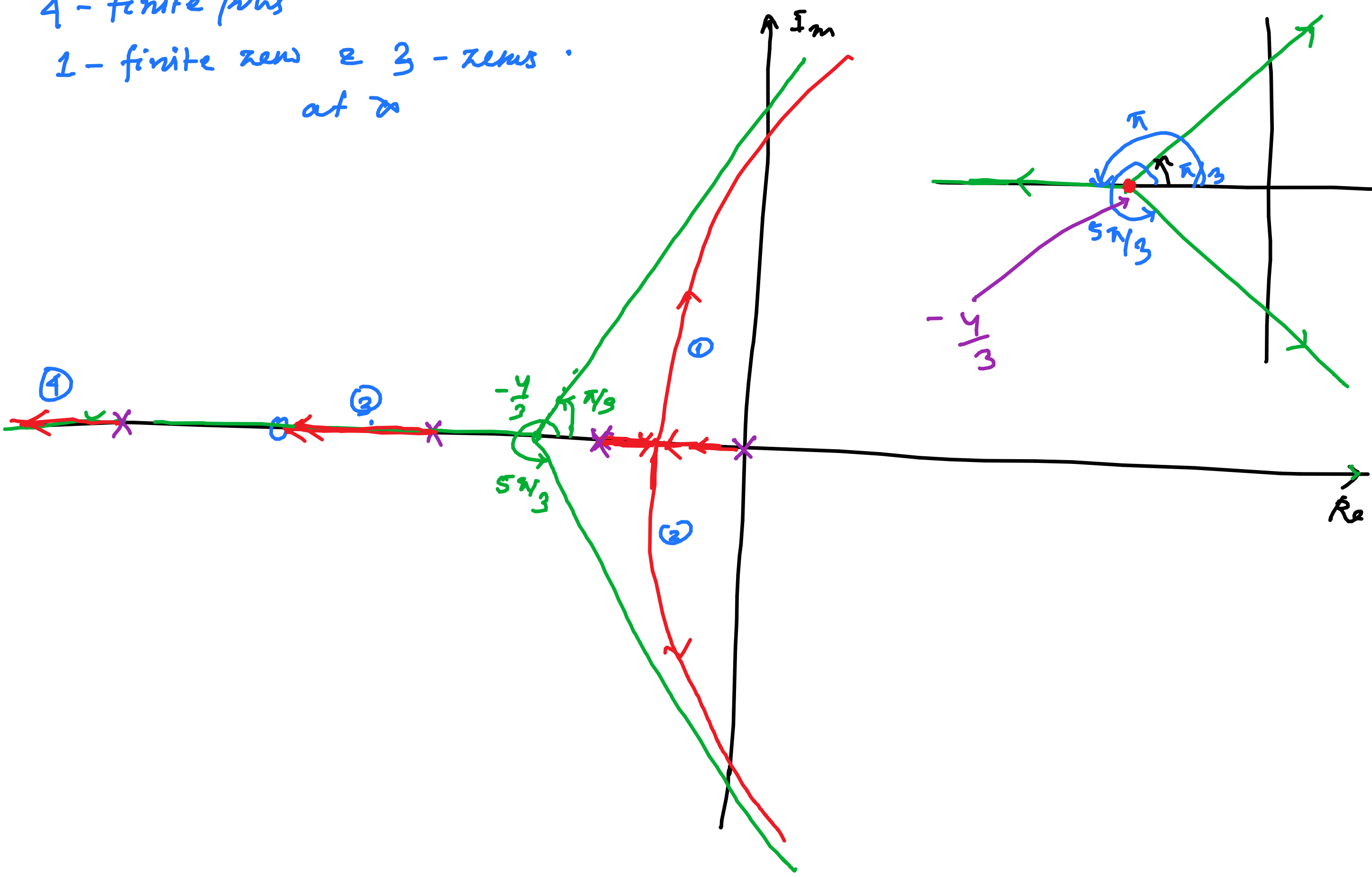
$= \pi/3$ for

$= \pi$ for $l = 1$

$= 5\pi/3$ for $l = 2$

4 - finite poles

1 - finite zero & 3 - zeros at ∞



Proof

Define $p :=$ no. of finite poles $-$ no. of finite zeros

$$p = n - m \quad \rightarrow \quad n = p + m$$

Let us write

$$K\,G(s)\,H(s) = \frac{K\,b(s) \leftarrow m^{th} \text{ deg.}}{a(s) \leftarrow n^{th} \text{ deg}} \qquad n \geqslant m$$

$$= \frac{K\left(s^m + b_1 s^{m-1} + \cdots + b_m\right)}{s^{p+m} + a_1 s^{p+m-1} + \cdots + a_{p+m}}$$

$$= \frac{K}{\left(\dfrac{s^{p+m} + a_1 s^{p+m-1} + \cdots + a_{p+m}}{s^m + b_1 s^{m-1} + \cdots + b_m}\right)}$$

$$s^m + b_1 s^{m-1} + \cdots + b_m \Big) \; s^{p+m} + a_1 s^{p+m-1} + \cdots + a_{p+m} \Big(\; s^p + (a_1 - b_1) s^{p-1} + \cdots$$

$$\underline{s^{p+m} + b_1 s^{p+m-1} + \cdots}$$

$$(a_1 - b_1)\, s^{p+m-1} + \cdots$$

$$\underline{(a_1 - b_1)\, s^{p+m-1} + \cdots}$$

$$\vdots$$

So

$$K\,G(s)\,H(s) = \frac{K}{s^p + (a_1 - b_1)\, s^{p-1} + \cdots}$$

The point 's' is on the root-locus if $K\,G(s)H(s) = -1$

Since we are interested in the behavior of Root-locus when $s \to \infty$, the terms s^p & s^{p-1} will dominate the smaller degree terms in (*)

- Hence with approximation, (*) can be represented as:

$$\underbrace{K\,G(s)\,H(s)}_{-1} \approx \frac{K}{s^p + (a_1 - b_1)s^{p-1}}$$

$$\Rightarrow s^p + (a_1 - b_1)\,s^{p-1} = -K$$

$$\Rightarrow s^p\left[1 + \frac{a_1 - b_1}{s}\right] = -K$$

Taking pth root

$$\Rightarrow s\underbrace{\left[1 + \frac{a_1 - b_1}{s}\right]^{1/p}}_{} = -K^{1/p} \quad \ldots \text{(**)}$$

$$(1 + x)^{n} \rightarrow \quad x = \frac{a_1 - b_1}{s} \qquad n = 1/p$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$\left[1 + \frac{a_1 - b_1}{s}\right]^{1/p} = 1 + \frac{1}{p}\left(\frac{a_1-b_1}{s}\right) + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}\left(\frac{a_1-b_1}{s}\right)^2 + \cdots$$

Since we are looking at $s \to \infty$, the terms $\frac{1}{s^2}, \frac{1}{s^3} \cdots$ become very small in comparison to $\frac{1}{s}$, & hence can be neglected.

$$\left[1 + \frac{a_1 - b_1}{s}\right]^{1/p} \approx 1 + \frac{a_1 - b_1}{ps}$$

From (**), we have following approximation

$$s\left[1 + \frac{a_1 - b_1}{ps}\right] = -K^{1/p}$$

$$\Rightarrow s + \frac{a_1 - b_1}{p} = -K^{1/p}$$

$$(-K)^{1/p} = \left(|K| e^{j\theta}\right)^{1/p} \quad \text{where } \theta = (2l+1)\pi$$

$$= |K|^{1/p} e^{j\theta_a} \qquad \theta_a = \frac{(2l+1)\pi}{p}$$

$$= |K|^{1/p}\left(\cos\theta_a + j\sin\theta_a\right)$$

by putting $s = \sigma + j\omega$

$$\sigma + j\omega + \frac{a_1 - b_1}{p} = |K|^{1/p}\left(\cos\theta_a + j\sin\theta_a\right)$$

By equating real & imaginary part

$$\begin{cases} \sigma + \dfrac{a_1 - b_1}{p} = |K|^{1/p} \cos\theta_a \\ \omega = |K|^{1/p} \sin\theta_a \end{cases}$$

by dividing

$$\frac{\omega}{\sigma + \frac{a_1 - b_1}{p}} = \frac{\sin\theta_a}{\cos\theta_a}$$

$$\Rightarrow \boxed{\omega = m(\sigma - \sigma_a)}$$

equation of straight lines

where

$$m = \tan\theta_a$$

$$\sigma_a = -\frac{a_1 - b_1}{p}$$

real axis intercept.

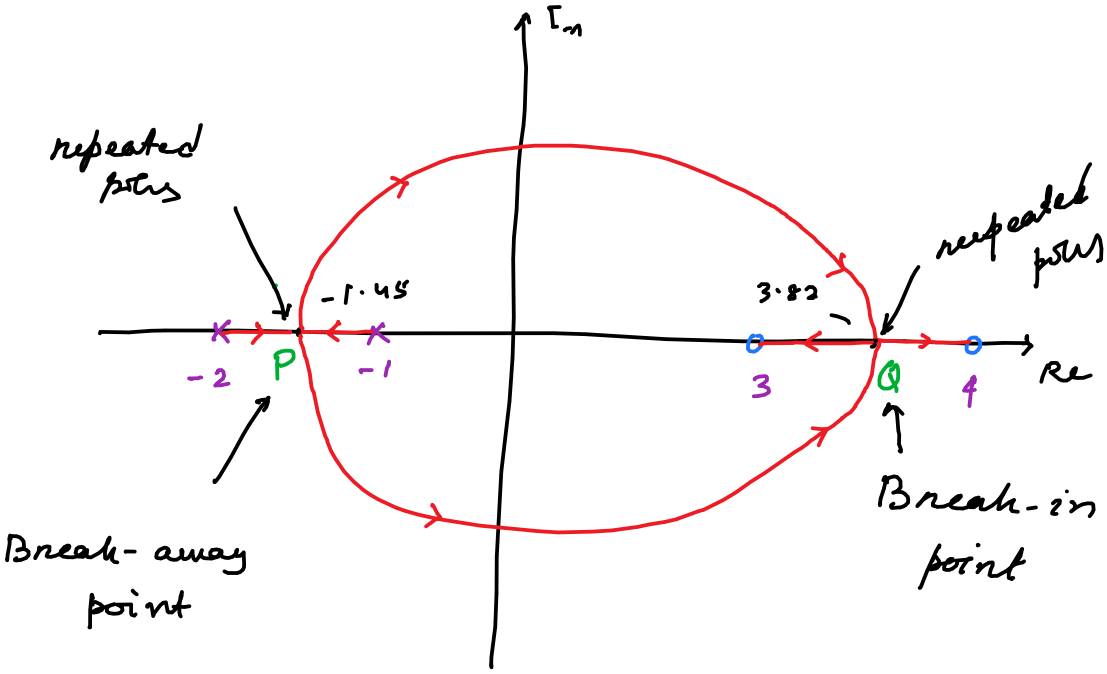
$$p = n - m$$

$$\theta_a = \frac{(2l+1)\pi}{p}$$

$$\begin{cases} a_1 = -\sum \text{finite poles} \\ b_1 = -\sum \text{finite zeros} \end{cases}$$

→ Real Axis Break-away & Break-in points

$$K\,G(s)H(s) = \frac{K\,(s-3)(s-5)}{(s+1)(s+2)}$$



Since equal no. of finite poles & zeros of $KG(s)H(s)$ there are no branches, which will go to ∞.

- The gain K is <u>maximum</u> <u>along the real axis</u> at the point P where break-away occurs.

- The gain K is <u>minimum</u> <u>along the real axis</u> at the point Q where break-in occurs.
  (K has to increase to reach zeros)

Since we are looking at maximum & minimum on the real axis, we can take $s = \sigma + j0$

$K G(\sigma) H(\sigma)$ ← Evaluate this quantity

Since P & Q are on the root locus

$$K G(\sigma) H(\sigma) = -1$$

$$\Rightarrow K = \frac{-1}{G(\sigma) H(\sigma)}$$ (K is a funct of σ)

Then to compute Break-away & Break-in points find σ from following event

$$\frac{dK}{d\sigma} = 0$$

then find σ

$$K G(s) H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

$$\frac{K(\sigma^2 - 8\sigma + 15)}{\sigma^2 + 3\sigma + 2} = -1$$

$$\Rightarrow K = \frac{-(\sigma^2 + 3\sigma + 2)}{\sigma^2 - 8\sigma + 15}$$

$$\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{(\sigma^2 - 8\sigma + 15)^2} = 0$$

$$\Rightarrow \sigma = -1.45 \ \& \ 3.82$$