ELL225

Lecture-6

- Non-linear Systems ← No transfer function

$\downarrow \quad \dot{x} = f(x, u)$

Linearize

$$\dot{x} = f(x)$$

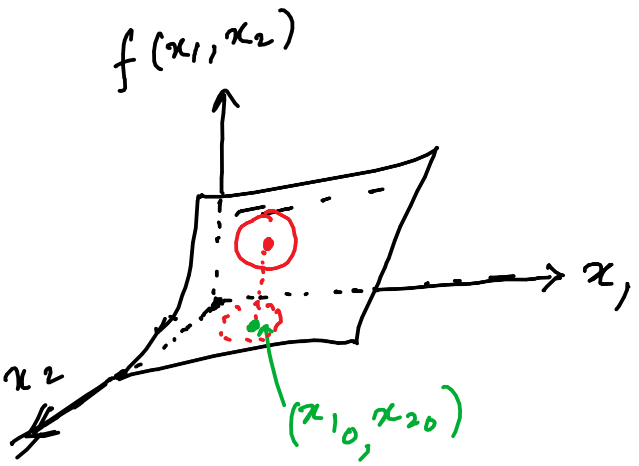
$$\downarrow$$

$$\Delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x_0} \Delta x + f(x_0)$$

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

← Multivariate functions where the system state variables are greater than 1.



$$\Delta x_1 = x_1 - x_{10}$$

$$\Delta x_2 = x_2 - x_{20}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Taylor series expansion of $f_1(x_1, x_2)$ around the point (x_{10}, x_{20})

$$f_1(x_1, x_2) = f_1(x_{10}, x_{20}) + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{10}, x_{20})} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{10}, x_{20})} \Delta x_2$$

$$f_2(x_1, x_2) = f_2(x_{10}, x_{20}) + \left.\frac{\partial f_2}{\partial x_1}\right|_{x_0} \Delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{x_0} \Delta x_2$$

$$\Delta \dot{x}_1 = \dot{x}_1 = f_1(x_1, x_2)$$

$$\Delta \dot{x}_2 = \dot{x}_2 = f_2(x_1, x_2)$$

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}_{x_0}}_{\text{A} \leftarrow \text{real matrix}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} f_1(x_0) \\ f_2(x_0) \end{bmatrix}}_{\text{real vector}}$$

$$\dot{x} = f(x) \;\downarrow\; \Delta\dot{x} = A\,\Delta x + h$$

Jacobian Matrix

$$\dot{x}_1 = f_1(x_1, x_2, u)$$

$$\dot{x}_2 = f_1(x_1, x_2, u)$$

linearize (x_0, u_0)

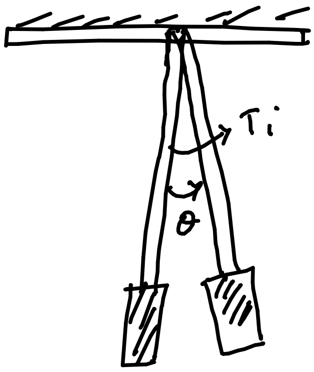
$$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\Delta \dot{x}_1 = f_1(x_0, u_0) + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_0,u_0)} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_0,u_0)} \Delta x_2 + \left.\frac{\partial f_1}{\partial u}\right|_{(x_0,u_0)} \Delta u$$

$$\Delta \dot{x}_2 = f_2(x_0, u_0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_0,u_0)} \Delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_0,u_0} \Delta x_2 + \left.\frac{\partial f_2}{\partial u}\right|_{(x_0,u_0} \Delta u$$

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(x_0,u_0)}}_{A} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{(x_0,u_0)}}_{B} \Delta u + \underline{\underline{\begin{bmatrix} f_1(x_0, u_0) \\ f_2(x_0, u_0) \end{bmatrix}}}$$

↓ Zero vecto when linearized about an equilibrium pt.



$$x = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}$$

No input $T_i = 0$

$$\dot x_1 = x_2$$

$$\dot x_2 = -\frac{D}{ml^2} x_2 - \frac{g}{l}\sin x_1$$

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2) = -\frac{D}{ml^2} x_2 - \frac{g}{l}\sin x_1$$

We need to linearize about the point $x_0 = (\pi/4, 0)$

$$\begin{bmatrix} \Delta \dot x_1 \\ \Delta \dot x_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} f_1(x_0) \\ f_2(x_0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{D}{ml^2} \end{bmatrix}}_{(\pi/4,\,0)} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} f_1(x_0) \\ f_2(x_0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cdot\frac{\sqrt2}{2} & -\frac{D}{ml^2} \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{g}{l}\cdot\frac{\sqrt2}{2} \end{bmatrix}}_{= h}$$

$(\pi, 0)$

$$\begin{bmatrix} 0 & 1 \\ g/l & -D/ml^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Delta \dot x = A \Delta x$$

- Linearize about Equillibrium Points
  on
  Steady State operating points

$$\frac{dx}{dt} = 0$$

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\} \rightarrow \left.\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}\right\}$$

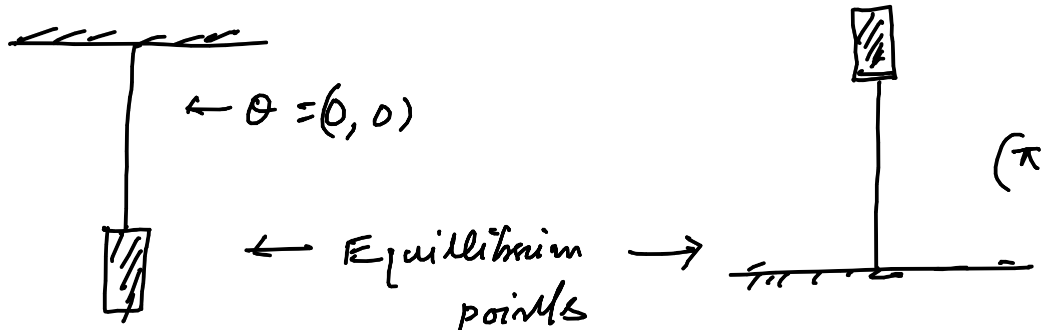
Non-linear algebraic equation

Solve the set of nonlinear algebraic equations
&
find the solution x_0, which is an equillibrium point.

$$\dot{x}_1 = x_2 \qquad\qquad x_2 = 0$$

$$\dot{x}_2 = -\frac{D}{ml^2} x_2 + \frac{g}{l} \sin x_1 \qquad\qquad x_1 = n\pi$$

when $n = 0, 1, 2 \ldots$



$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\} \longrightarrow x_0 \rightarrow \begin{aligned} f_1(x_{10}, x_{20}) &= 0\\ f_2(x_{10}, x_{20}) &= 0\end{aligned}$$

A linaized model ("linearized" about equilibrium point)

$$\boxed{\dot{x} = A\Delta x}$$

$$\dot{x}_1 = f_1(x_1, x_2, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u)$$

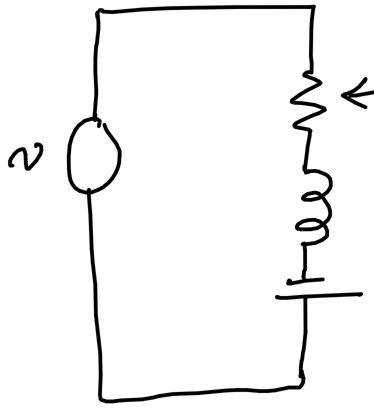
- A point (x_0, u_0) is an equilibrium point if there is a specific input u_0 s.t.

$$f(x_0, u_0) = 0$$

$$\dot{x}_1 = f_1(x_1, x_2, \cdots x_n, u_1, u_2 \cdots u_m)$$
$$\dot{x}_2 = f_2(x_1, x_2 \cdots x_n, u_1, u_2 \cdots u_m)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2 \cdots x_n, u_1, u_2 \cdots u_m)$$
$$\downarrow$$

linearize about equilibm point

$$\boxed{\Delta\dot{x} = A\Delta x + B\Delta u}$$



← Non linear resistor $i_R = 2e^{0.1 v_R}$ $L = 1$

$$\frac{di}{dt} = -10 \ln i/2 + 20 + v$$

$$-10 \ln i/2 + 20 = 0 \quad \text{(No input } v = 0\text{)}$$

$$\Rightarrow \ln i/2 = 2 \quad \Rightarrow i = 2e^2 = 14.7\,A$$

$(x_0, u_0) = (14.7, 0)$ ← equillibrium point

If $v = 10\,V$

$$\ln i/2 = 3 \quad \Rightarrow \quad i = 2e^3 = 40.17$$

$$(x_0, u_0) = (40.17, 10)$$

$$\frac{di}{dt} = -10 \ln i/2 + 20 + v$$

$$\Delta \dot{i} = -10 \left. \frac{1}{i} \right|_{(14.7, 0)} \Delta i + 1\,\Delta v$$

$$\Delta \dot{i} = -0.68\,\Delta i + \Delta u$$

So T.F. $\dfrac{\Delta i(s)}{\Delta u(s)}$ = Can be evaluated.