ELL225

Lecture-9

- System Analysis

- All signals are assumed to be Causal signals (starts at $t=0$)

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \text{ state ~~sfm~~ }$$

$$G(s) = \frac{Y(s)}{U(s)} \quad \Big\} \text{ Transfer funct}^n \text{ representation}$$

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Impulse function

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = g(t)$$

Impulse response of a system

$$f(t)\,\delta(t) = f(0)\,\delta(t)$$

$$f(t)\,\delta(t-T) = f(T)\,\delta(t-T)$$

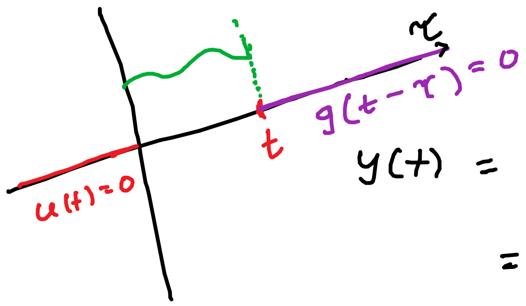
$$u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = u(t) * g(t)$$

input signal

Convolution operation

impulse response

- Signals are causal $\Rightarrow$ $u(t) = 0 \quad t < 0$
- System is also causal $\Rightarrow$ there will be no response of the system before the start of input signal $u(t)$



$$y(t) = u(t) * g(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)\, g(t-\tau)\, d\tau \quad \leftarrow \text{ the time axis is in } \tau$$

$$= \int_{0}^{\infty} u(\tau)\, g(t-\tau)\, d\tau \quad \leftarrow \text{ Since } u(t) \text{ is causal} \Downarrow u(t) = 0 \quad t < 0$$

Since $g(t-\tau)$ is causal

$g(t-\tau) = 0$ for $t - \tau < 0$

$\Rightarrow \tau > t$

$$= \int_{0}^{t} u(\tau)\, g(t-\tau)\, d\tau$$

$$y(t) = u(t) * g(t)$$

Laplace transform

$$Y(s) = G(s)\,U(s)$$

$\mathcal{L}(\delta(t)) = 1$

$\mathcal{L}(u_s(t)) = 1/s$

unit step function

for $u(t) = \delta(t)$ $\quad = G(s)$

for unit step input $\quad Y(s) = \frac{1}{s} \cdot G(s)$

$Y(s) = G(s)$ for impulse input

let $G(s) = \frac{1}{s^2+3s+2}$

$$y(t) = \mathcal{L}^{-1}(G(s))$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right)$$

$\downarrow$ Partial fract' expansion

$$= \mathcal{L}^{-1}\left(\frac{A_1}{s+2} + \frac{A_2}{s+1}\right)$$

$$= A_1 e^{-2t} + A_2 e^{-t}$$

real coefficient polynomial

- T. F. $\quad G(s) = \frac{b(s)}{a(s)}$ ← another polynomial.

- Poles : The values of s for which the

$\times$ t.f. $G(s)$ becomes ∞.

(the roots of $a(s)$)

• Zeros : The value of s when $G(s)$ becomes 0.
(the roots of $b(s)$)

$$\delta(t) \longrightarrow \boxed{G(s) = \frac{1}{s^2+3s+2}} \longrightarrow y(t) = A_1 e^{-2t} + A_2 e^{-t}$$

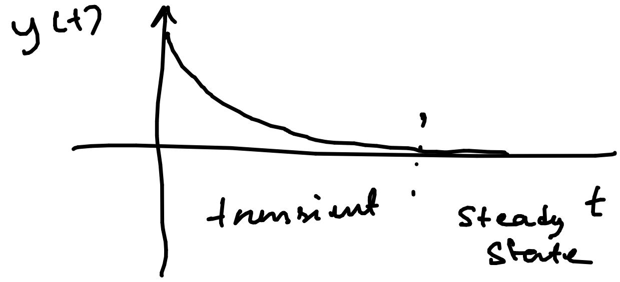
$\underline{\underline{t \geq 0}}$

The time response depends on which parameters of the t.f $G(s)$?

• $s^2+3s+2 = (s+2)(s+1)$

-2 & -1 are the poles of the syst.

• Time response of a system depends on the poles of a system.



• The roots of $a(s)$ (possible cases)

(i) real & distinct roots

(ii) real & repeated roots

(iii) Complex conjugate pair roots

(iv) repeated complex conjugate pairs

Example

$$G(s) = \frac{2}{(s+1)(s+2)^2} \qquad \text{real \& repeated}$$

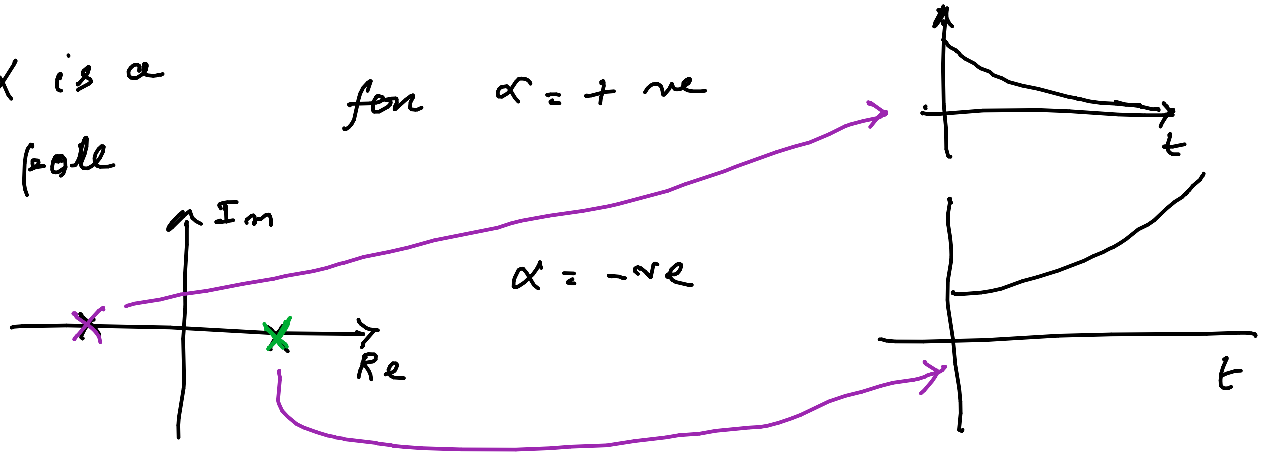
$$= \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{A_3}{(s+2)^2}$$

$$\mathcal{L}^{-1} \qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$A_1 e^{-t} \qquad\quad A_2 e^{-2t} \qquad\quad \underline{\underline{A_3 t e^{-2t}}}$$

$$\frac{1}{s+\alpha} \rightarrow y(t) = e^{-\alpha t}$$

α is a pole



- ## Case - 3

Distinct complex conjugate pair :

$$\mathcal{L}\left[A e^{-at} \cos \omega t\right] = \frac{A(s+a)}{(s+a)^2 + \omega^2}$$

$$\mathcal{L}\left[A e^{-at} \sin \omega t\right] = \frac{A\omega}{(s+a)^2 + \omega^2}$$

$$\mathcal{L}\left[A e^{-at} \cos \omega t + B e^{-at} \sin \omega t\right] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2}$$

$$G(s) = \frac{3}{s(s^2+2s+5)}$$

Poles $s = 0$

$s = -1 \pm 2j$

$$= \frac{3/5}{s} - \frac{3}{5}\,\frac{s+2}{s^2+2s+5}$$

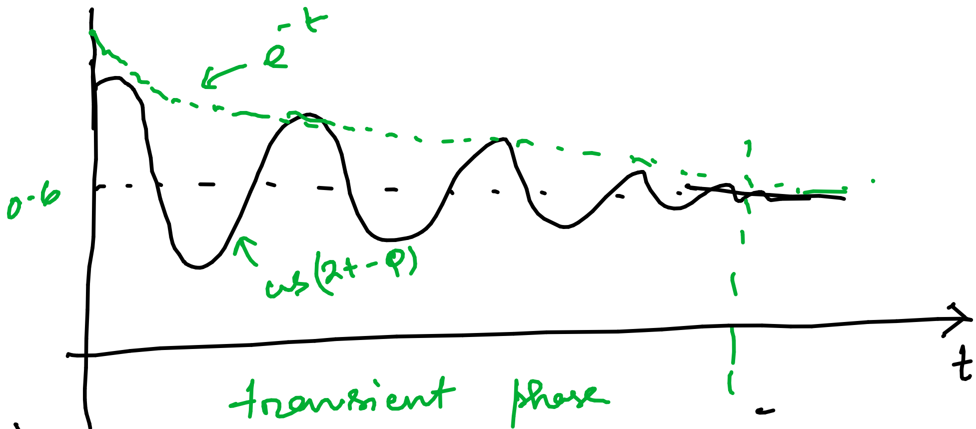
$$= \frac{3/5}{s} - \frac{3}{5}\,\frac{(s+1)+(1/2)\cdot 2}{(s+1)^2+2^2}$$

$$= \underbrace{3/5}_{s=0} - \underbrace{\frac{3}{5}e^{-t}\left(\cos 2t + \tfrac{1}{2}\sin 2t\right)}_{s=-1+2j}$$

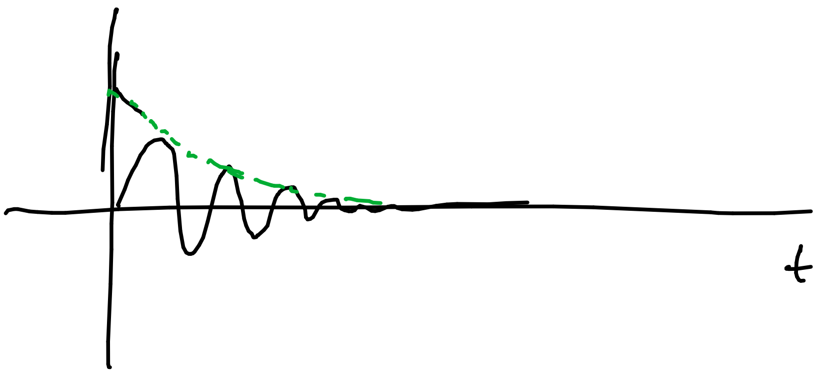
$$= 0.6 - 0.671\, e^{-t}\cos(2t - \varphi)$$

$$\cos\varphi = \frac{1}{\sqrt{1+1/2^2}} \qquad \sin\varphi = \frac{1/2}{\sqrt{1+(1/2)^2}}$$

$t = \infty$



$e^{-5t}\cos(2t-\varphi)$



Poles on Roots of $a(s)$	$G(s)$	$g(t)$
(i) Distinct real roots	$\frac{A}{s+a}$	Ae^{-at} $\quad \underline{t \geq 0}$
(ii) Real repeated roots	$\frac{A}{(s+a)^r}$	$\frac{A t^{r-1}}{(r-1)!} e^{-at}$
(iii) Distinct complex conjugate pair	$\frac{b(s)}{s^2+as+b}$	$A e^{-at} \cos(bt-\varphi)$
(iv) Repeated complex conjugate pair	$\frac{b(s)}{(s^2+as+b)^2}$	$\frac{A t^{r-1}}{(r-1)!} e^{-at} \cos(bt-\varphi)$

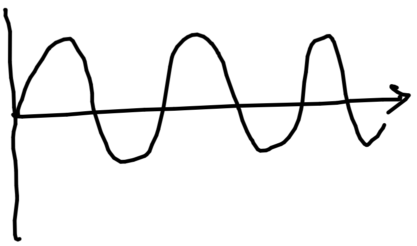
The second order system

$$\boxed{G(s) = \frac{b}{s^2 + as + b}}$$

← Written in this standard fn.

If $a = 0$ $\quad G(s) = \frac{b}{s^2 + b}$

$s = \pm j\sqrt{b}$



Natural frequency: $\omega_n = \sqrt{b}$

Represent → $s = \sigma \pm j\omega$

Damping ratio : $\frac{\text{Exponential decay freq.}}{\text{Natural freq.}} = \frac{|\sigma|}{\omega_n}$

ξ $\quad = \frac{a/2}{\omega_n}$

$$G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- Rise time
- Peak time
- Settling time
- Maximum over shoot