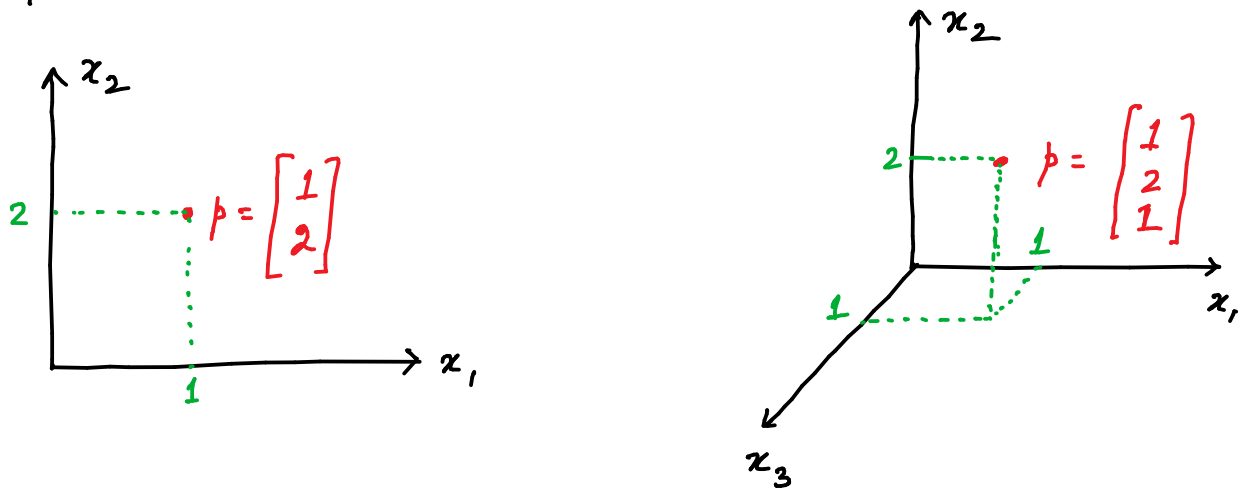


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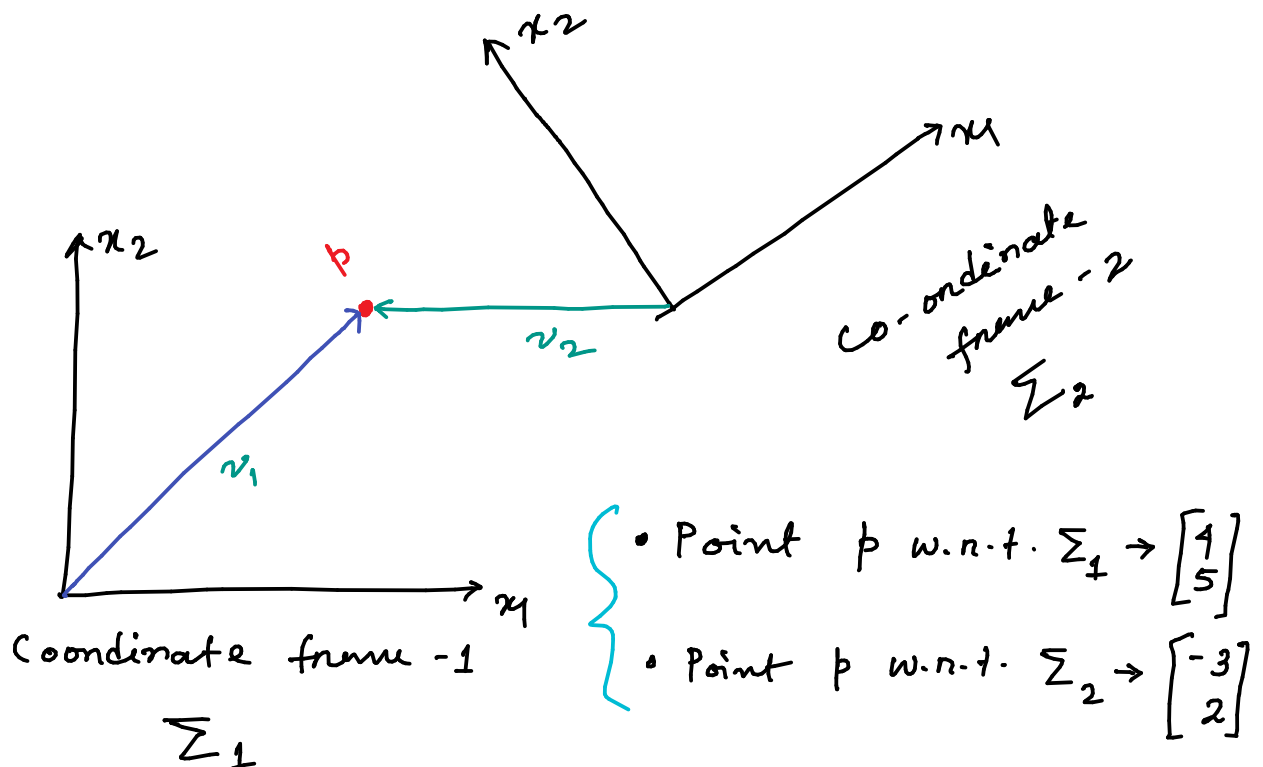
RIGID MOTION

Part - 1

- Representation of a point in Euclidean space :



- Representation of a point w.r.t. different co-ordinate frames :



- Corresponding to the point p , we have two vectors

$$(i) \quad v_1 \text{ in } \Sigma_1$$

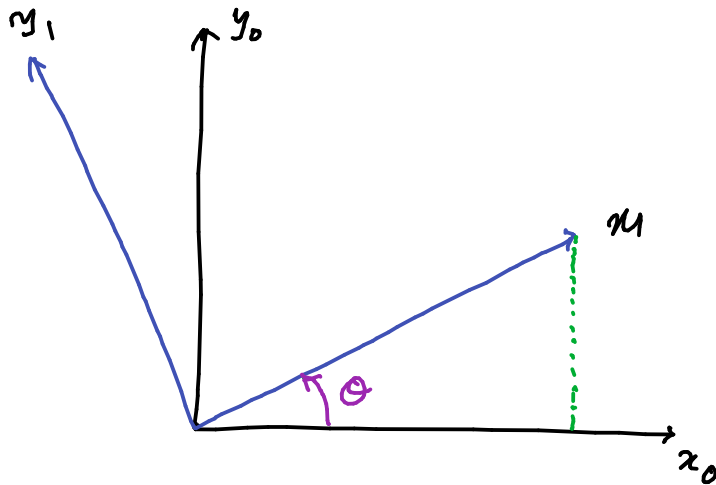
$$(ii) \quad v_2 \text{ in } \Sigma_2$$

- We represent a vector same as a point, i.e.

$$v_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{in } \Sigma_1$$

$$v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{in } \Sigma_2$$

→ Rotation in a Plane:



Consider the following two coordinate frames:

(i) Σ_0 with axes x_0, y_0

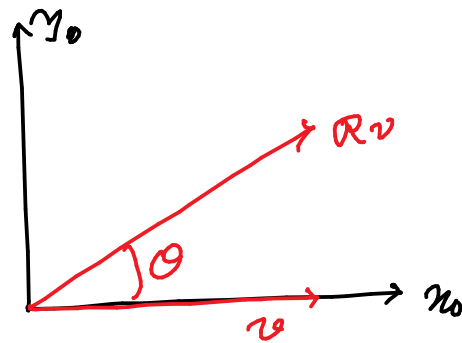
(ii) Σ_1 with axes x_1, y_1

Σ_1 is obtained by rotating Σ_0 by an angle θ .

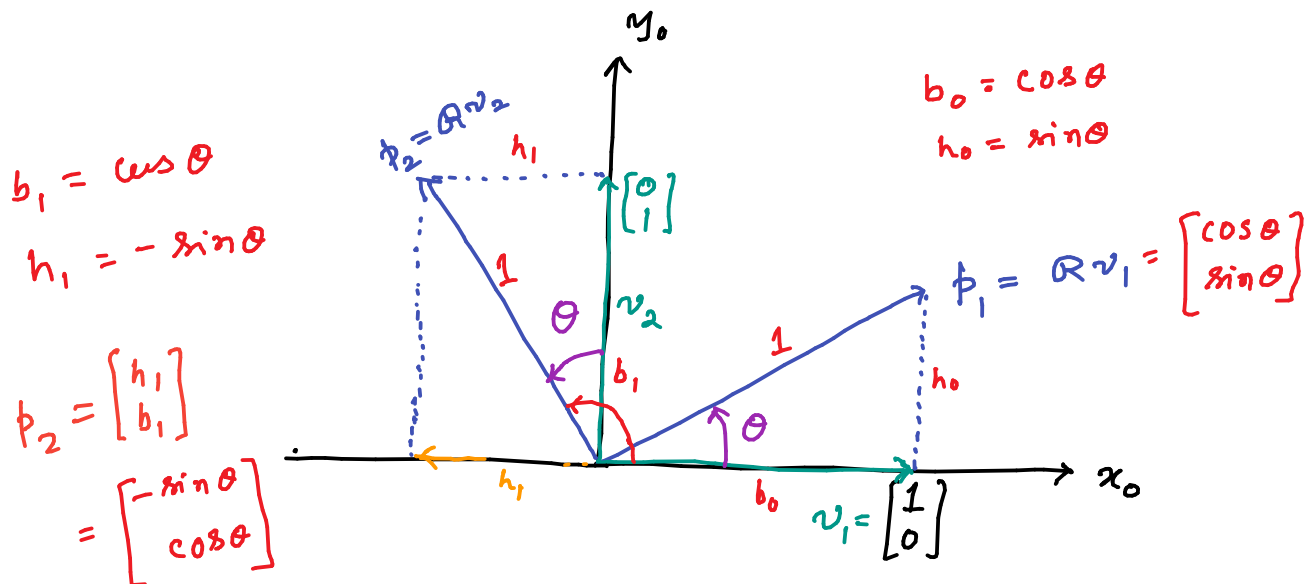
* An operation that rotates a vector v by an angle θ can be represented as a matrix R . For 2-D case

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

Then $v \xrightarrow{R} Rv$
 $\uparrow$ obtained by rotation of v
 by an angle θ as in fig



let us now see what are the entries of R ?



Consider the coordinate reference frame

Σ_0 . Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be a unit vector
 in Σ_0 & it is along x_0 .

$$v_1 \xrightarrow{R} Rv_1$$

Note that

$$p_1 = Rv_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

Since $\|v_1\| = 1$ (unit length), after rotation of this vector, the magnitude must remain same.

$\Downarrow$

$$\|Rv_1\| = \cos^2\theta + \sin^2\theta = 1$$

$\Downarrow$

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$R \quad v_1$

$\Downarrow$

$$\begin{bmatrix} r_{11} \\ r_{12} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Similarly $p_2 = Rv_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

Note that $\|Rv_2\| = 1$

$\Downarrow$

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Hence the operator R that rotates a vector in Σ_0 -frame is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

↑ called "Rotation Matrix"
in 2-D plane.

* Note that the elements of R are obtained by representing the axis vectors (x_1, y_1) of coordinate frame Σ_1 with $\begin{matrix} \parallel & \parallel \\ p_1 & p_2 \end{matrix}$ respect to co-ordinate frame Σ_0 .

|||

i.e. by assigning the coordinates to p_1 & p_2 w.r.t. Σ_0 (x_0 - y_0 axes).

↓

The coordinate frame Σ_0 is referred to as "Reference Frame".

→ Some Properties of Rotation Matrix :

- A matrix Q is said to be "orthogonal"

if $\boxed{Q^T Q = I} \Rightarrow \boxed{Q^{-1} = Q^T}$

- The rotation matrix is orthogonal.

$$\begin{aligned} R^T R &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

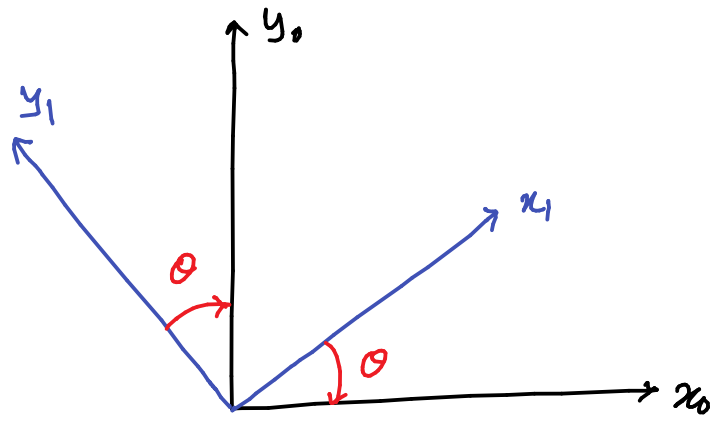
- The determinant of $R = \pm 1$

$$\boxed{\det(R) = \pm 1}$$

- The 2×2 of such matrices (Orthogonal) is referred to as

"Special Orthogonal Group
of order 2"

Notation $\rightarrow SO(2)$



If we assume that Σ_1 frame (x_1, y_1 axis) is reference frame, then Σ_0 frame is obtained by θ° clock-wise rotation of Σ_1 -frame. Hence, the associated rotation matrix can be obtained directly by replacing " θ " with " $-\theta$ ".

↓

$$R_0 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \xrightarrow{\theta \rightarrow -\theta} R_1 = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

||

$$\boxed{R_0^T = R_0^{-1}}$$

→ Assume that the coordinate frame Σ_1 is obtained by rotating Σ_0 -frame by an angle θ in counter-clockwise direction.



- If we consider Σ_0 -frame as reference frame, then the vectors of Σ_1 -frame can be obtained by multiplying the rotation matrix

$$R_0 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- If Σ_1 -frame is considered as reference frame, then the vectors of Σ_0 -frame can be obtained by multiplying the rotation matrix

$$R_1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

- We also have the following result:

$$R_1 = R_0^T = R_0^{-1}$$

→ The notation in 2-D can be extended to 3-D by assuming that a plane (2-D) is rotated about the third axis.

For instance :

x_0-y_0 plane is rotated about the

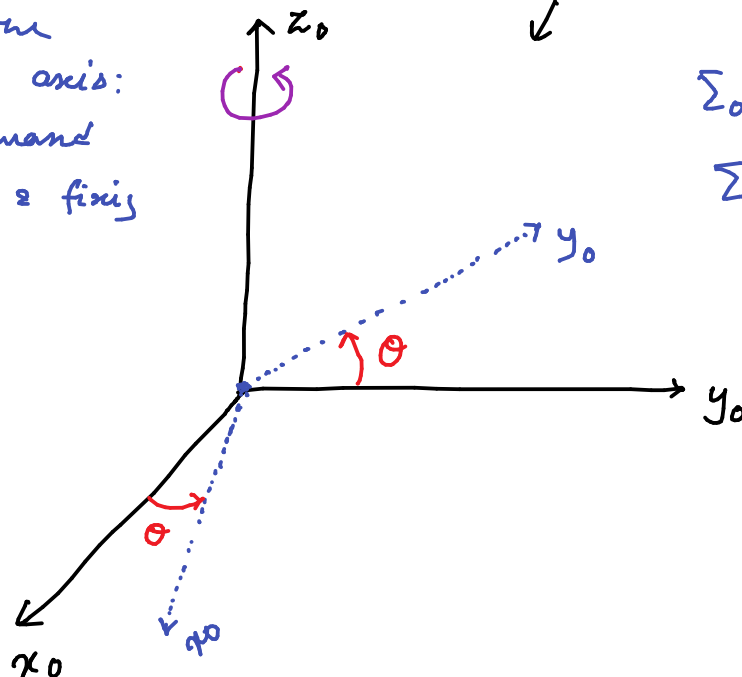
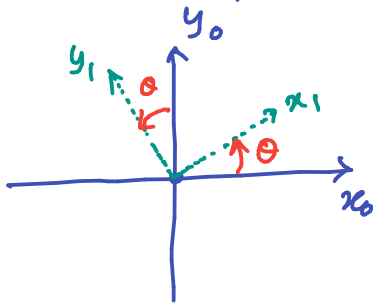
z_0 axis to obtain x_1-y_1

↓

$$\Sigma_0 \rightarrow (x_0, y_0, z_0)$$

$$\Sigma_1 \rightarrow (x_1, y_1, z_0)$$

One can apply the rotation rule in the following co-ordinate axis:
(by bringing z_0 forward directⁿ to outward & fixing at origin)



Σ_1 is obtained by rotating the

x_0-y_0 plane by an angle θ , in

counterclockwise direction, about the axis z_0 .

⇓

The rotation matrix must not alter the z -component.

For, a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ w.r.t. Σ_0 -frame,

the component v_3 must not alter

for $R_0 v$, whereas other

two components v_1 & v_2 should be

obtained by rotation. Hence R_0 can be

of this form:

$$R_0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↓ Rotation

R_0^z

- Note that R_0^z is also an orthogonal matrix & it satisfies $\det(R_0^z) = \pm 1$.

↓

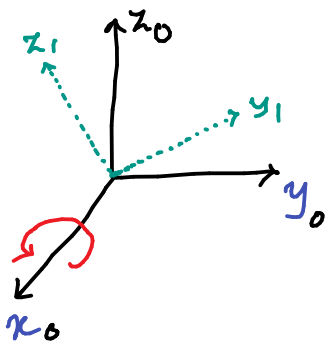
Hence

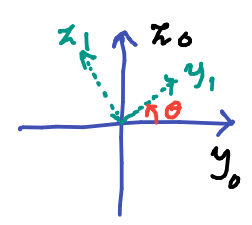
$$R_0^z \in SO(3)$$

- If we consider Σ_1 as reference, then

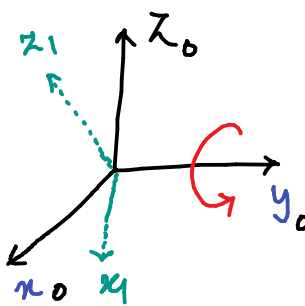
$$R_1^z = (R_0^z)^T = (R_0^z)^{-1}$$

- The rotation matrix for rotation about x -axis (y - z plane rotation in counter clockwise) :



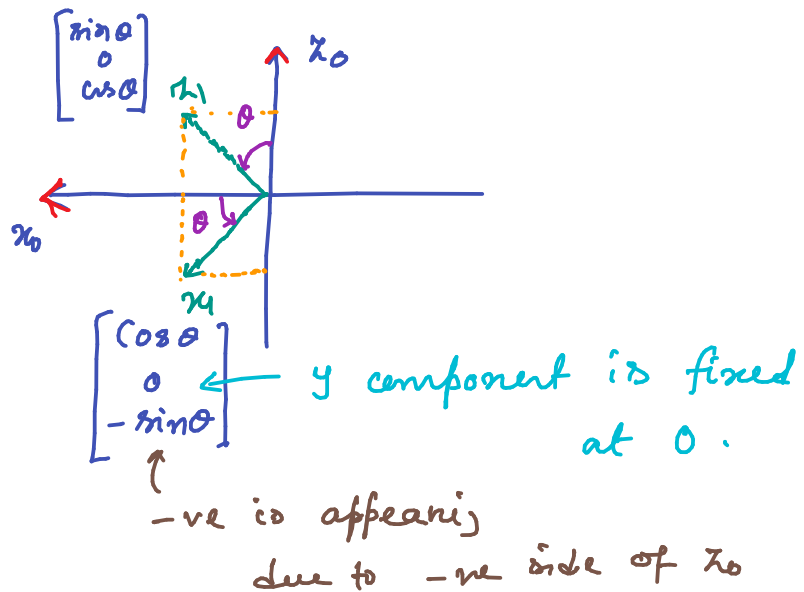
$$R_0^x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$


- Rotation matrix, for rotation about y -axis:



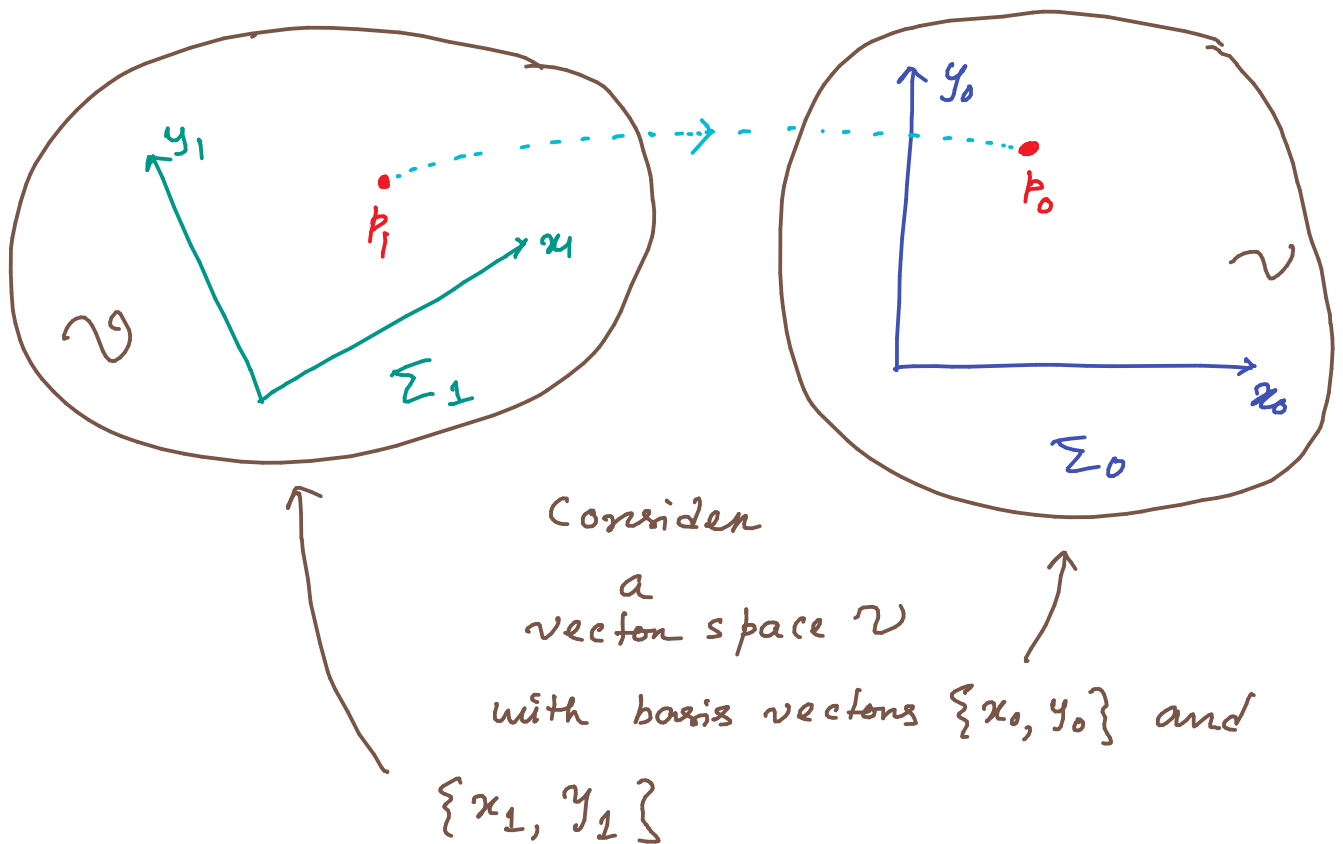
$$R_0^y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

bringing y_0 forward lined' to outwards & fixing it at origin, we have the following coordinate axes:



→ Rotational Transformation

↑ Transform the co-ordinates of a point from one coordinate frame to another.



- Given a point p with its coordinates $p_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ w.r.t. basis $\{x_1, y_1\}$, we need to know the coordinates of same point p i.e. p_0 w.r.t. the basis $\{x_0, y_0\}$.

↓

$$\text{let } p_0 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{ w.r.t. } \{x_0, y_0\}$$

Then the following relation holds:

$$\underbrace{\alpha_1 x_1 + \alpha_2 y_1}_{p_1} = \underbrace{\beta_1 x_0 + \beta_2 y_0}_{p_0}$$

$p_1 \quad p_0$

$\searrow \quad \swarrow$
 p same point representⁿ
w.r.t. different basis.

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ x_1 & y_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_0 & y_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_0 & y_0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ x_1 & y_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \dots (*)$$

Now for our case, the basis vectors are

$$x_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad y_1 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \quad \left. \vphantom{\begin{matrix} x_1 \\ y_1 \end{matrix}} \right\} \text{Represent } \Sigma_1$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad y_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \left. \vphantom{\begin{matrix} x_0 \\ y_0 \end{matrix}} \right\} \text{Represent } \Sigma_0$$

Σ_1 is obtained by counterclockwise rotation of Σ_0 by an angle θ .

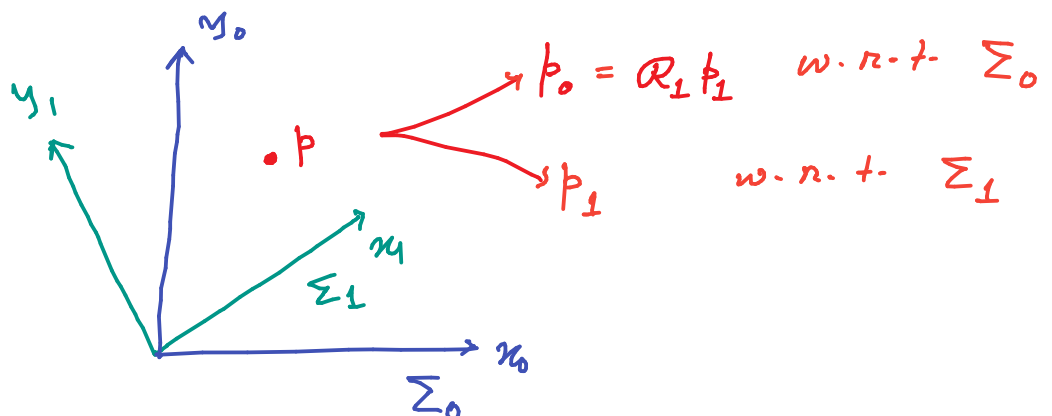
Then from the relation $\textcircled{*}$, we have

$$\underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{p_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R_1} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{p_1}$$

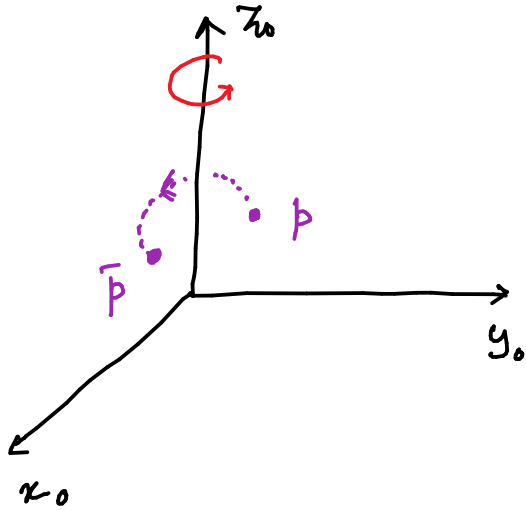
$$\Rightarrow \boxed{p_0 = R_1 p_1} \quad \leftarrow \text{This is the relation}$$

how the coordinates
of a point p w.r.t. different
frames i.e. $p_1 \in p_0$, are
related.

* If a given point p is expressed
w.r.t. Σ_1 by coordinates p_1 , then
 $p_0 = R_1 p_1$ represent the same point p , expressed
w.r.t. different frame Σ_0 .



Example



The original point p is rotated about z -axis in counter clockwise direction by an angle 180° to obtain new point $\bar{p}$.

Assume that a new frame Σ_1 is obtained by counterclockwise rotation of Σ_0 about z_0 -axis.



The associated rotation matrix is

$$R_1 = \begin{bmatrix} \cos \pi & -\sin \pi & 0 \\ \sin \pi & \cos \pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Given the coordinates of point p , we need to find the coordinates of $\bar{p}$.



Assume that a coordinate frame Σ_1 , axes aligned with Σ_0 , is rigidly attached with the point p , whose coordinates are given.

→ Now rotate the coordinate frame Σ_1 by an angle π . Then, the point p will also be rotated by an angle π & the new point is $\bar{p}$.

↓

Note that $\bar{p}$ will have the same coordinates as p w.r.t. the frame Σ_1 .

↓

Now to obtain the new coordinates of $\bar{p}$ w.r.t. Σ_0 , we use the previous rule:

$$\bar{p} = R_1 p \quad \text{let } p = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$