

ELL 406
RIGID MOTION

Part - 2

→ Composition of Rotations

Consider following three coordinate frame with common origin.

$$\left. \begin{array}{l} \Sigma_0 \rightarrow (x_0, y_0, z_0) \\ \Sigma_1 \rightarrow (x_1, y_1, z_1) \\ \Sigma_2 \rightarrow (x_2, y_2, z_2) \end{array} \right\} \text{One is obtained from other through rotation.}$$

Let p be a point, whose coordinates are described w.r.t. Σ_0, Σ_1 & Σ_2 are as follows :

$$\left. \begin{array}{l} p^0 \leftarrow \text{w.r.t. } \Sigma_0 \\ p^1 \leftarrow \text{w.r.t. } \Sigma_1 \\ p^2 \leftarrow \text{w.r.t. } \Sigma_2 \end{array} \right\} \begin{array}{l} \text{corresponds} \\ \text{to} \\ \text{same point} \\ p \end{array}$$

→ Recall : If p is a point with its coordinates p^1 w.r.t. Σ_1 , then its coordinates p^0 w.r.t. Σ_0 is

$$\boxed{p^0 = R_1 p^1} \dots \textcircled{1} \quad \text{where } R_1 \text{ is the rotation matrix associated with obtaining } \Sigma_1 \text{ from } \Sigma_0.$$

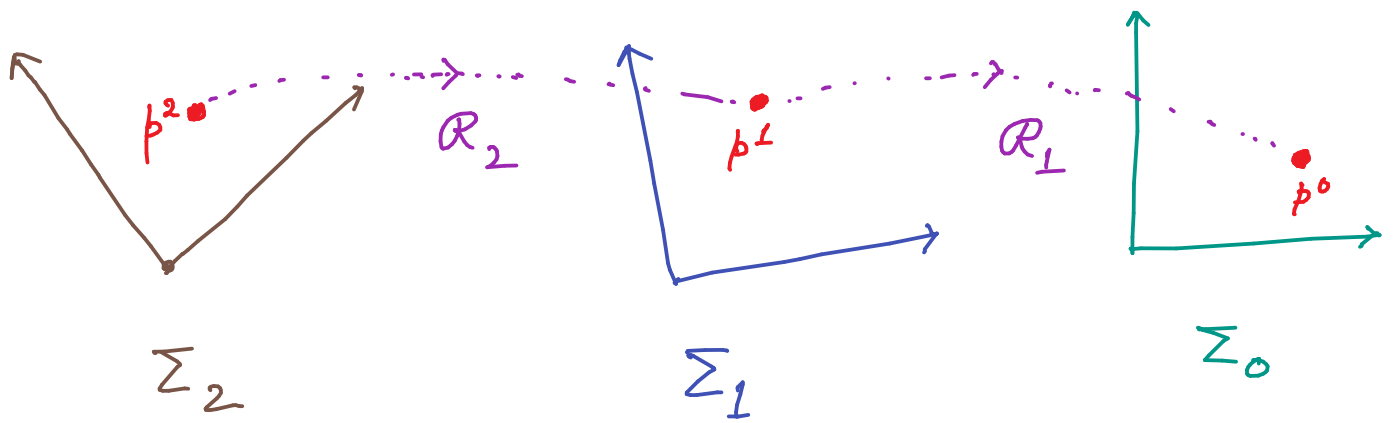
• Similarly if p^2 is coordinates of point p w.r.t. the coordinate frame Σ_2 , then its coordinates p^1 w.r.t. coordinate frame Σ_1 is :

$$\boxed{p^1 = R_2 p^2} \dots \textcircled{2}$$

• Assuming p^2 be the coordinates of p w.r.t. frame Σ_2 , let p^0 be the coordinates of p , w.r.t. the coordinate frame Σ_0 .

Then

$$\boxed{p^0 = R_0 p^2} \dots \textcircled{3}$$



[Note that Σ_0 , Σ_1 & Σ_2 have a common origin & each one is obtained through rotation of another. They are shown in different figures just for understanding.]

We have the following relation:

$$\begin{aligned} p^1 &= R_2 p^2 \\ p^0 &= R_1 p^1 \end{aligned}$$

R_i : rotation matrices

$$\begin{aligned} p^0 &= R_1 p^1 \\ &= R_1 R_2 p^2 \end{aligned} \quad \leftarrow \text{comparing with (3)}$$

$\underbrace{R_1 R_2}_{R_0}$

$$R_0 = R_1 R_2 \quad \leftarrow \text{Composition law for rotational transformations.}$$

* R_1 & R_2 are not commutative, & hence order of multiplication is important.

* The composition law $R_0 = R_1 R_2$ can be interpreted as follows:

To transform the coordinates of the point p from its representation p_2 w.r.t. the coordinate frame Σ_2 to its representation p^0 w.r.t. Σ_0 -frame, we

- first transform $p^2 \rightarrow p^1$ w.r.t.

frame Σ_1 using the rotation matrix R_2 ,

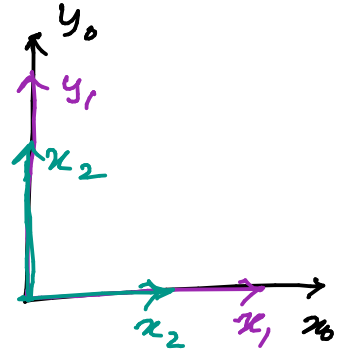
- then transform $p^1 \rightarrow p^0$ using the rotation matrix R_1 .

$$\begin{array}{c} p^2 \xrightarrow{R_2} p^1 \xrightarrow{R_1} p^0 \\ R_0 = R_1 R_2 \end{array}$$

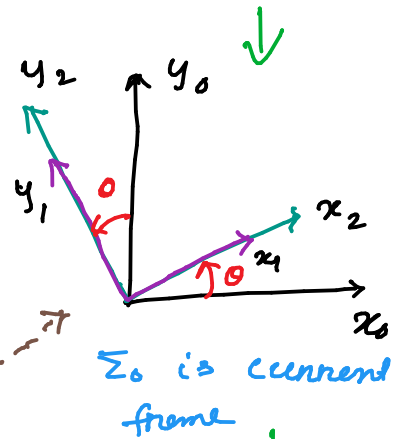
→ Second interpretation of Composition of Rotation matrices:

$$R_0 = R_1 R_2$$

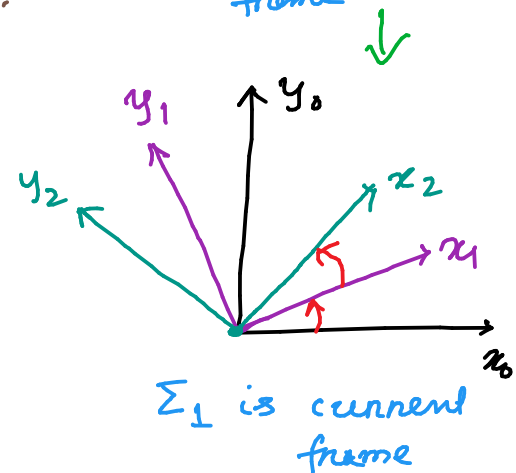
- let all three coordinate frames Σ_0 , Σ_1 & Σ_2 coincide. →



↓
First rotate Σ_2 (with Σ_1) relative to Σ_0 according to transformation matrix R_1



↓
With the frames Σ_2 & Σ_1 coincident, rotate Σ_2 w.r.t. Σ_1 using the rotation matrix R_2 .



- * Remember the order of rotation & the multiplication of rotation matrices!

- * In each rotation, the coordinate frame with respect to which the rotation occurs is called "Current Frame".

Example

Let R represents a rotation of angle ϕ about the current y -axis followed by a rotation of angle θ about current z -axis. Then R is ?

- First rotation about y -axis through an angle ϕ . The associated rotation matrix is

$$R_{y\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

- Second rotation is about z -axis through an angle θ . The associated rotation matrix:

$$R_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence } R = R_{y\phi} R_{z\theta} = \begin{bmatrix} C_\phi C_\theta & -C_\phi S_\theta & S_\phi \\ S_\phi C_\theta & C_\phi & 0 \\ -S_\phi C_\theta & S_\phi S_\theta & C_\phi \end{bmatrix}$$

$\cos\theta \rightarrow C_\theta \quad \sin\theta \rightarrow S_\theta$

→ Rotation about Fixed Frame:

- In the Current Frame, we obtain new coordinate frames by successive rotation with respect to the current frame, which was newly generated through previous rotation.
- In the Fixed Frame, we obtain the new coordinate frames successively through rotation with respect to the original/initial frame, which is referred to as Fixed frame.

(We do not consider the newly generated frame in previous rotation as reference)

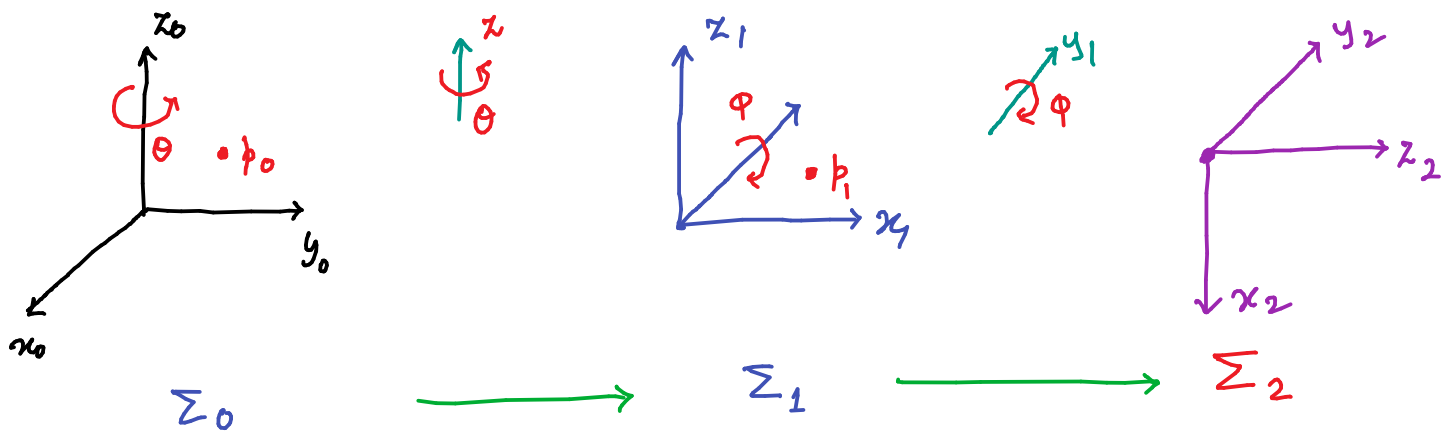
→ Using basis change approach, let us first redefine the composition rule in the "Current Reference" frame work.



consider the sequence of rotations

(i) Rotation Θ about z -axis

(ii) Rotation Φ about y -axis



let basis for $\Sigma_0 \rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis for $\Sigma_1 \rightarrow \left\{ \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} -s_\theta \\ c_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\Rightarrow \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_0 \\ p_0 \end{bmatrix}$$

$$\Rightarrow p_0 = R_1 p_1 \dots \dots \textcircled{1}$$

- For the next notation, Σ_1 is considered as reference frame. Then Σ_2 is obtained through rotation w.r.t. Σ_1 .

$\Downarrow$

The basis for Σ_1 can now be considered as $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ & basis for Σ_2 is

$$\left\{ \begin{bmatrix} \cos \phi \\ 0 \\ -\sin \phi \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sin \phi \\ 0 \\ \cos \phi \end{bmatrix} \right\}$$

$\Downarrow$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \\ p_1 \end{bmatrix}$$

$$\Rightarrow p_1 = R_2 p_2 \quad \dots \quad (2)$$

Combining (1) & (2)

$$\begin{aligned} p_0 &= R_1 p_1 \\ &= R_1 R_2 p_2 \\ &\quad \underbrace{\hspace{1cm}}_{R_0} \end{aligned}$$

- * For easy reading of Rotation matrices, we use following notation:

$$R_i^j : \text{Rotation matrix of } \Sigma_i \text{ w.r.t. } \Sigma_j$$

For instance, in previous case:

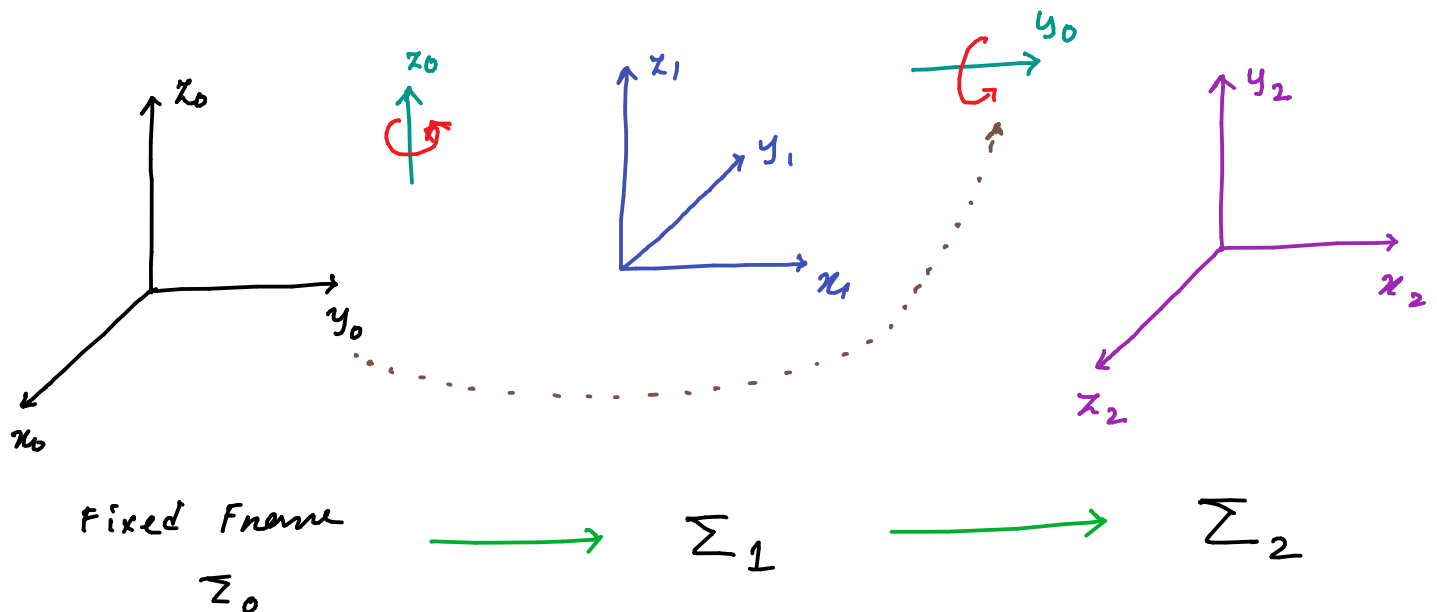
$$R_1 \rightarrow R_1^0 \quad R_2 \rightarrow R_2^1$$

Then

$$p_0 = \underbrace{R_1^0 R_2^1}_{R_2^0} p_2$$

* The reference frame basis are set to unit basis, whereas the rotated frame basis are the columns of associated rotation matrix.

→ We use this approach to obtain composition relation, when rotations are made with respect to a "Fixed Frame", not with the "Current Frame".



* In the previous figure following rotations are made w.r.t. fixed frame Σ_0

(i) Rotation about z_0 -axis $\rightarrow \Sigma_1$

(ii) Rotation about y_0 -axis $\rightarrow \Sigma_2$

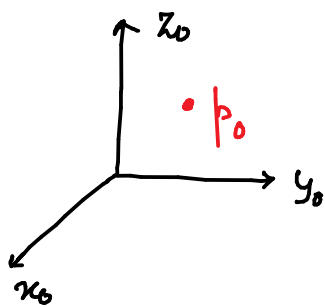
↓
i.e. rotation of Σ_1
about the axis which is
aligned with y_0 , i.e. x_1

↓ (current frame approach)

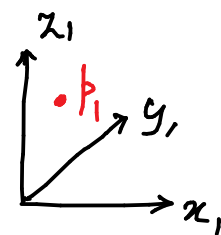
This says that, the same composition Rule we can not apply for this case!

→ To obtain the composition matrix, we consider the following steps:

• Step - 1



basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$



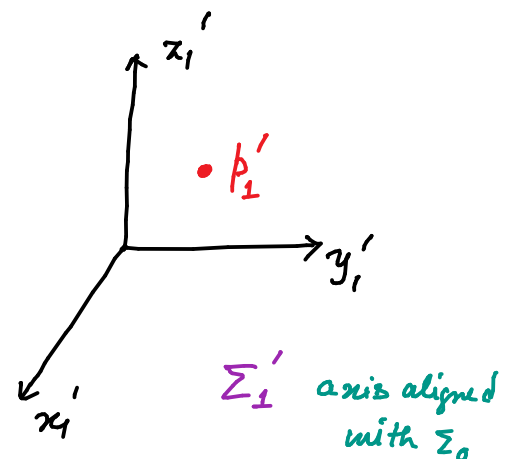
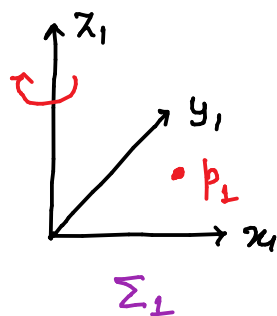
basis $\left\{ \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} -s_\theta \\ c_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\boxed{p_0 = R_1^0 p_1} \dots \dots \textcircled{1}$$

From this point onwards, we give a sequence of operations/rotations so that previous current frame approach can be used. For this we set the sequence of frames as current frames, about which rotation is made to obtain next frame.

• Step-2

Since the rotation is not done w.r.t Σ_1 to obtain Σ_2 , let us first reorient Σ_1 s.t. its axes aligned with Σ_0 .



is considered as current frame

& its basis are set to unit basis

basis

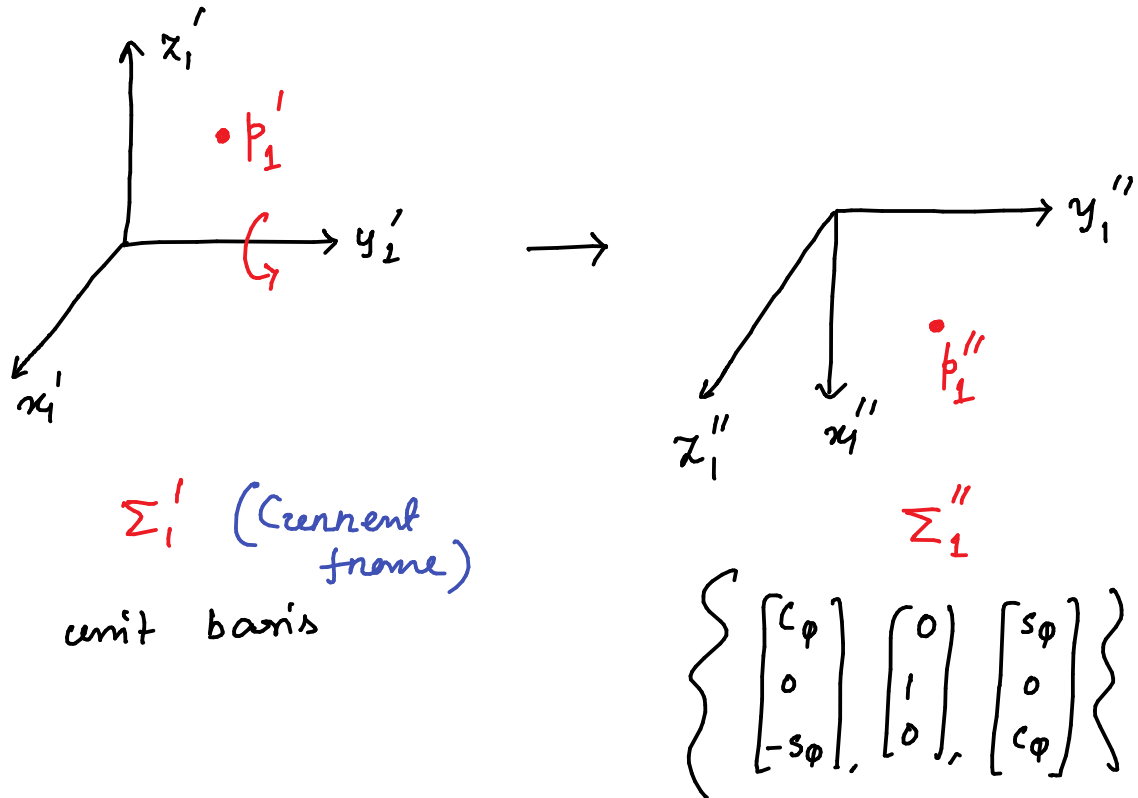
$$\left\{ \begin{bmatrix} c_\theta \\ -s_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} s_\theta \\ c_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\boxed{p_1 = R_{1'}^1 p'_1} \dots \textcircled{2}$$

$$\boxed{R_{1'}^1 = (R_1^0)^T}$$

- Step-3

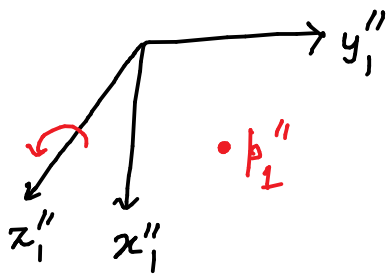
Give rotation to Σ_1' same as rotation needs to be given to obtain Σ_2 from Σ_1 , i.e. rotation about y_0 -axis.



$$\boxed{p_1' = R_2 p_1''} \dots \textcircled{3} \quad R_{1''}^{1'} = R_2$$

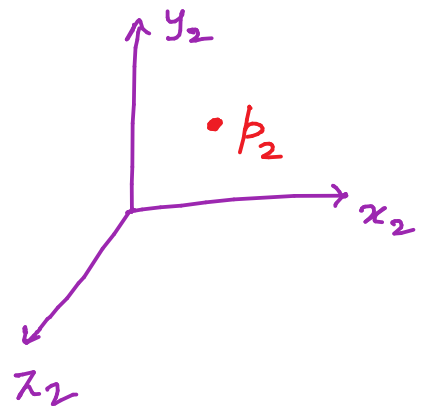
- Step-4

We now need to give the same rotation to Σ_1'' , as we had given Σ_0 to obtain Σ_1 , i.e. counter clockwise rotation to Σ_1'' about z_1'' axis.



Σ_1''

current frame, whose basis can be set as unit basis



Σ_2

$$\left\{ \begin{bmatrix} c_{\theta} \\ s_{\theta} \\ 0 \end{bmatrix}, \begin{bmatrix} -s_{\theta} \\ c_{\theta} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\boxed{p_1'' = R_2^{1''} p_2} \dots \textcircled{4}$$

→ We have now following set of $\textcircled{4}$ equations:

$$\left. \begin{aligned} p_0 &= R_1^0 p_1 \\ p_1 &= R_{1'}^1 p_1' \\ p_1' &= R_2 p_1'' \\ p_1'' &= R_2^{1''} p_2 \end{aligned} \right\} \rightarrow$$

$$\begin{aligned} p_0 &= R_1^0 p_1 \\ &= R_1^0 R_{1'}^1 p_1' \\ &= R_1^0 R_{1'}^1 R_2 p_1'' \\ &= R_1^0 R_{1'}^1 R_2 R_2^{1''} p_2 \end{aligned}$$

Note: We have the

$$\text{following relations: } R_{1'}^1 = (R_1^0)^T$$

2

$$R_2^{1''} = R_1^0$$

Then we have :

$$p_0 = R_1^0 (R_1^0)^T R_2 R_1^0 p_2$$

$$p_0 = \underbrace{R_2 R_1^0}_{R_2^0} p_2$$

R_2^0 the composition matrix.

* Notice the composition matrix that we had obtained with current frame & fixed frame.

- For current frame, some rotation matrix is post-multiplied with R_1^0 .
- For Fixed frame, some rotation matrix is pre-multiplied with R_1^0 .

In the fixed frame, the rotation

matrix R_2 is essentially same as

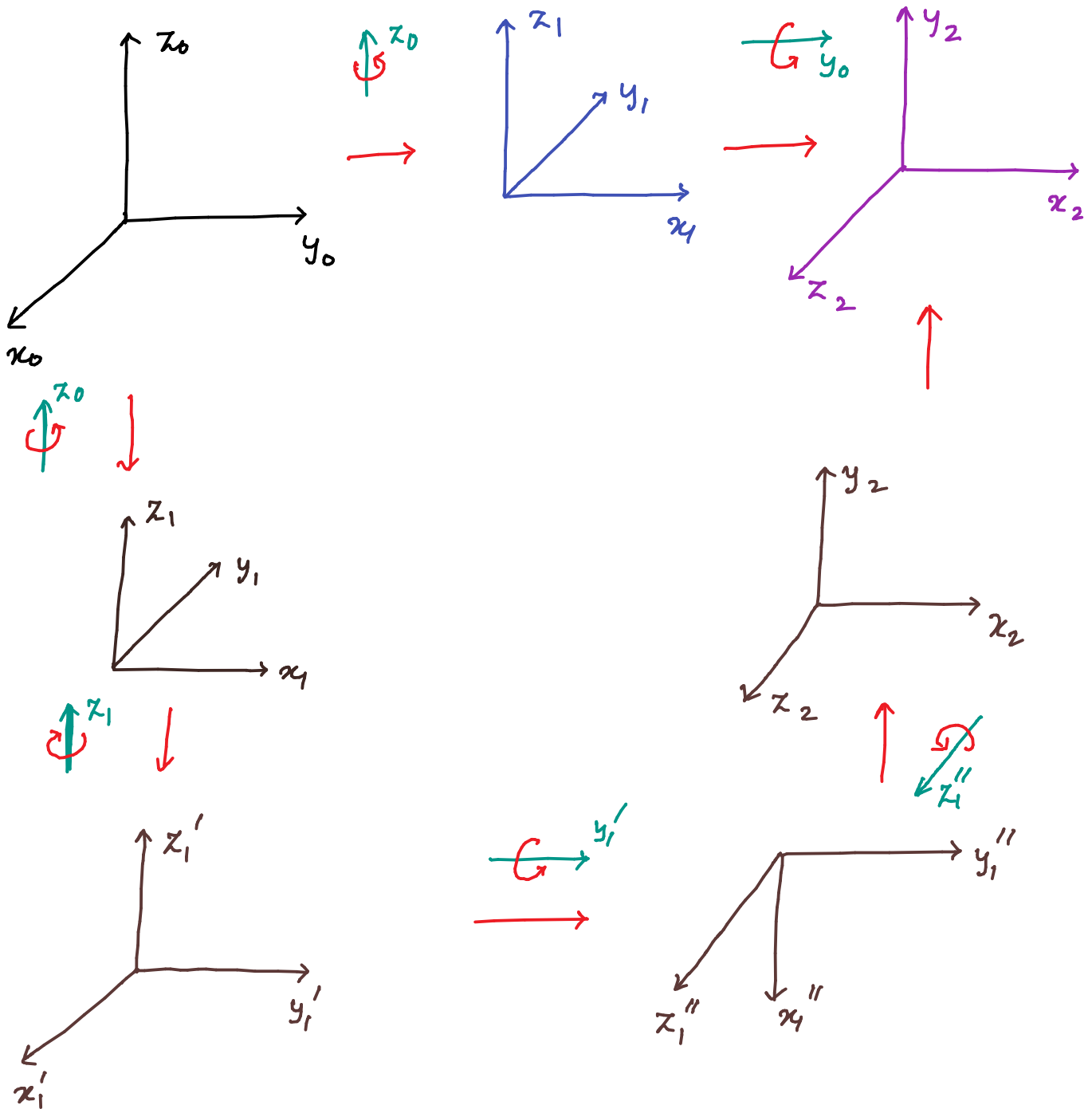
R_2^0 i.e. the rotation matrix

needed for 2nd rotation i.e.

about y_0 -axis of

Σ_0 -frame.

→ Overall Diagram for Fixed Frame



Example

Let following sequence of rotations are made :

- (i) rotation by θ about current x -axis
- (ii) rotation by ϕ about current z -axis
- (iii) rotation by α about fixed z -axis
- (iv) rotation by β about current y -axis.

What would the overall rotation matrix?

- Let Σ_0 be a coordinate frame.

rotation by θ about x -axis

↳ Rotation matrix R_θ^x

We begin with R_θ^x & keeps on pre- and post-multiplying rotation matrices accordingly.

$$R = R_\alpha^z R_\theta^x R_\phi^z R_\beta^y$$