

ELL 406

RIGID MOTION

Part - 3

- A rigid body possesses at most three rotational degrees of freedom.



At most three quantities are required to represent its orientation.

- We discuss a couple of procedures where arbitrary rotations can be represented using only three independent quantities:

- Euler-angle representation
- Roll-pitch-yaw representation

→ Euler-angle Representation:

Euler angles are usually represented by (ϕ, θ, ψ) .

- One can reach to any target co-ordinate frame by successively performing three elemental rotations, i.e. about the axes of a co-ordinate system.



- Let $\Sigma_0 (x_0, y_0, z_0)$ be a coordinate frame & $\Sigma_1 (x_1, y_1, z_1)$ be a target coordinate frame.



We can specify the orientation of Σ_1 w.r.t. Σ_0 through the Euler angles (ϕ, θ, ψ) by following three successive rotations:

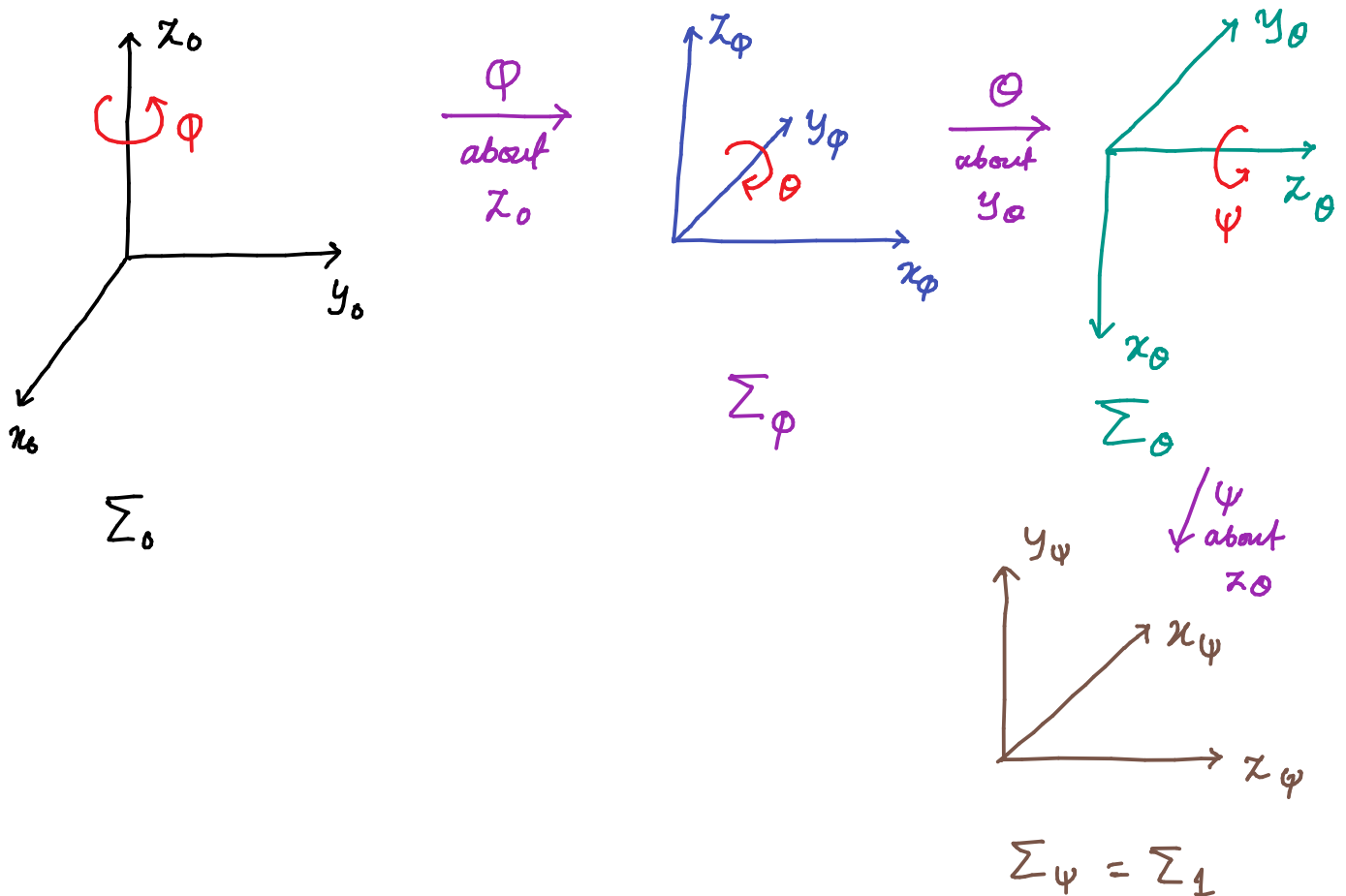
1. Rotate about the z_0 -axis by an angle ϕ .

↳ say the generated frame is Σ_ϕ
(current frame)

2. Rotate the current frame Σ_ϕ about the axis y_ϕ by an angle Θ .

↳ New (current) coordinate frame is Σ_Θ .

3. Rotate the current frame Σ_Θ about z_Θ -axis by an angle Ψ .



• For Euler-angle representation any one of the following sequence of three elementary rotations can be considered:

$z-y-z, x-z-x, y-x-y, z-x-z, x-y-z$

In the above, we have used $x-y-z$ sequence of elementary rotations.

- The above 3-rotations can be represented through the associated elementary rotation matrices:

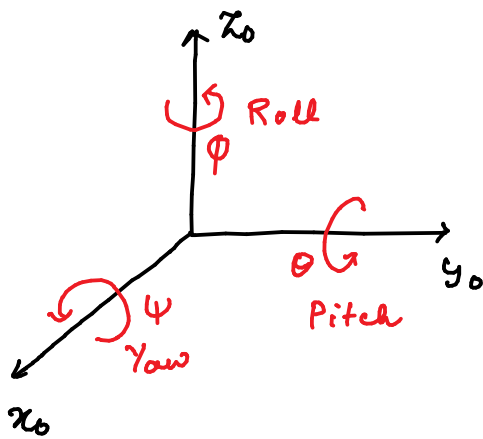
$$\begin{aligned} R_{zyx} &= R_{z\phi} R_{y\theta} R_{x\psi} \\ &= \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & -S_\psi & 0 \\ S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{bmatrix} \end{aligned}$$

↑ called ZYX -Euler-angle

Transformation Matrix.

- This matrix will change based on the sequence of rotations we consider.

→ Roll, Pitch, Yaw Angles Representation



In this representations
the three independent
angle of rotations are

ψ : Yaw angle

θ : Pitch angle

ϕ : Roll angle

With this representation, we consider following
three sequence of rotations:

1. Rotation about z_0 -axis by angle ϕ
to get desired roll.
2. Rotation about y_0 -axis by angle θ
to get desired pitch.
3. Rotation about x_0 -axis by an angle ψ
to get desired yaw.

Then the corresponding Roll-Pitch-Yaw
transformation matrix :

$$R_{RPY} := R_{z\phi} R_{y\theta} R_{x\psi}$$

$$R_{RPY} = R_{z\phi} R_{y\theta} R_{z\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

→ Rigid Motion

• Rigid motion refers to the combination of:

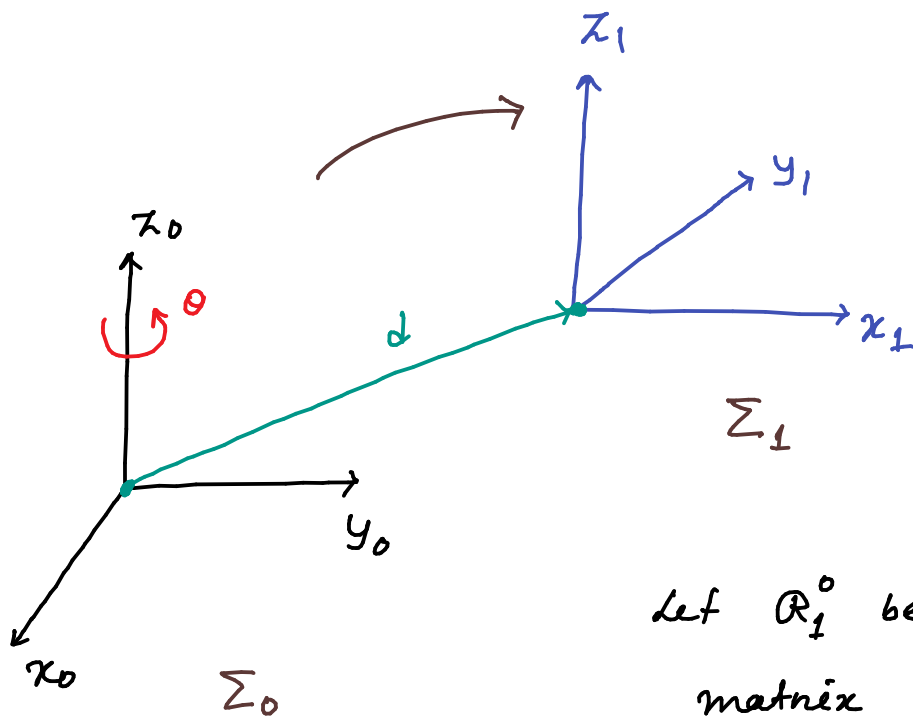
1. Translational Motion
2. Rotational Motion

• Definition: A 'rigid motion' is an ordered pair: (d, R) with $d \in \mathbb{R}^3$ & $R \in SO(3)$.

The group of all rigid motions is

referred to as "Special Euclidean Group",

denoted as $SE(3)$.



Let R_1^0 be the rotation matrix that specifies the orientation of Σ_1 with respect to Σ_0 .

d : the vector from the origin of Σ_0 to the origin of Σ_1

- Let p be a point which is rigidly attached to the coordinate frame Σ_1 . Let p_1 be its coordinates w.r.t. to the local frame Σ_1 .

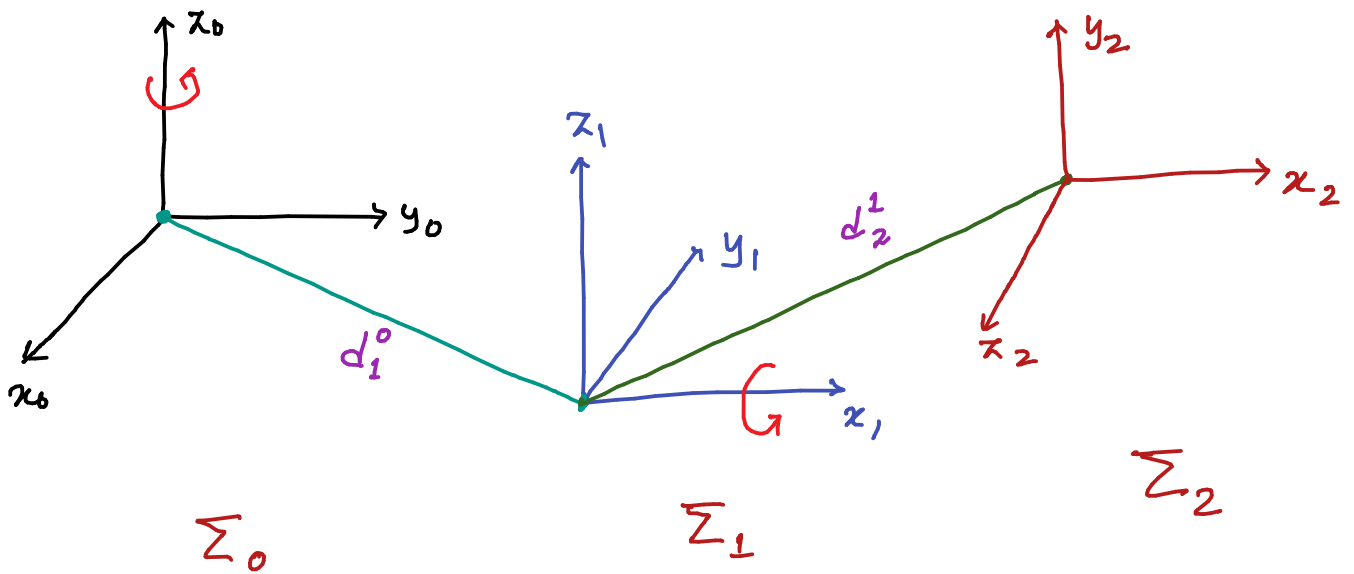


Then, the coordinates of p w.r.t. Σ_0

is p_0 , which is:

$$p_0 = R_1^0 p_1 + d_1^0$$

where d_1^0 is equal to d .



d_1^0 : The vector from origin of Σ_0 to the origin of Σ_1

d_2^1 : The vector from origin of Σ_1 to the origin of Σ_2

- Let a point p be rigidly attached with Σ_2 & its coordinates w.r.t. Σ_2 is p_2 .



Then the coordinates of p w.r.t. Σ_1 is:

$$p_1 = R_2^1 p_2 + d_2^1 \quad \dots \textcircled{1}$$

The coordinates of p w.r.t. Σ_0 is:

$$p_0 = R_1^0 p_1 + d_1^0 \quad \dots \textcircled{2}$$

Combining (1) & (2)

$$p_0 = R_1^0 p_1 + d_1^0$$

$$= R_1^0 (R_2' p_2 + d_2') + d_1^0$$

$$= \underbrace{R_1^0 R_2^1}_{R_2^0} p_2 + \underbrace{R_1^0 d_2^1 + d_1^0}_{d_2^0}$$

$$\beta_0 = R_2^0 \beta_2 + d_2^0$$

→ Representation of Composition of Rigid Motion
using Homogeneous Transformation :

• We have obtained

$$\begin{aligned} \mathcal{R}_2^0 &= \mathcal{R}_1^0 \mathcal{R}_2^1 \\ \mathcal{d}_2^0 &= \mathcal{R}_1^0 \mathcal{d}_2^1 + \mathcal{d}_1^0 \end{aligned}$$

}

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}}$$

- called "Homogeneous Transformation matrix" $\rightarrow H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$ when $R \in SO(3)$
 $d \in \mathbb{R}^3$

- A rigid motion can be represented through a Homogeneous transformation matrix

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

R is a rotation matrix

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

- To represent the coordinates of a point which is obtained through rigid motion, we can use the Homogeneous transformation matrix (HTM) H by augmenting the associated vectors with 4th element as 1 : For instance :

$$\bar{p}_0 = \begin{bmatrix} p_0 \\ 1 \end{bmatrix}$$

$$\bar{p}_1 = \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

Then

$$p_0 = R_1^0 p_1 + d_1^0 \quad \text{can be}$$

represented as follows:

$$\bar{p}_0 = \begin{bmatrix} p_0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}}_H \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

- Let H_1^0 be a HTM relating two coordinate frames Σ_0 & Σ_1 . Further let H be another HTM relating to Σ_1 & Σ_2 , where Σ_2 is obtained from the "Current Frame" Σ_1 through rigid motion then:

$$H_2^0 = H_1^0 H \quad \begin{array}{l} \text{(Post multiplication} \\ \text{of } H \text{ in Current} \\ \text{frame} \\ \text{approach)} \end{array}$$

- If Σ_2 is obtained w.r.t. the rigid motion of "Fixed Frame" Σ_0 then:

$$\boxed{H_2^0 = H H_1^0} \quad \begin{array}{l} \text{(Pre-multiplication} \\ \text{of } H) \end{array}$$

→ Some Basic Homogeneous Transformations

- Translation of 'a' unit in x direct?
(Only)

$$H_{xa}^T = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $H_{yb}^T = \begin{bmatrix} I & \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & 1 \end{bmatrix} \leftarrow \text{Only translation of 'b' unit in y-direction}$

- $H_{zc}^T = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & 1 \end{bmatrix} \leftarrow \text{Only translation of 'c' unit in z-direction.}$

- Only Rotation about x-axis with angle α :

$$H_{x\alpha}^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $H_{y\beta}^R = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Only rotation about y-axis with angle } \beta$

$$\bullet \quad H_{zr}^R = \begin{bmatrix} C_r & -S_r & 0 & 0 \\ S_r & C_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Rotation about } z\text{-axis} \\ \text{with angle } r.$$

→ Example

Following sequence of rigid motions are performed:

- Rotation by angle θ about current x -axis
- translation by 'a' unit along x -axis
- translation by 'd' unit along current z -axis
- rotation by angle β about current z -axis.

Then the associated HTM is

$$H = H_{x\theta}^R H_{xa}^T H_{zd}^T H_{z\beta}^R$$