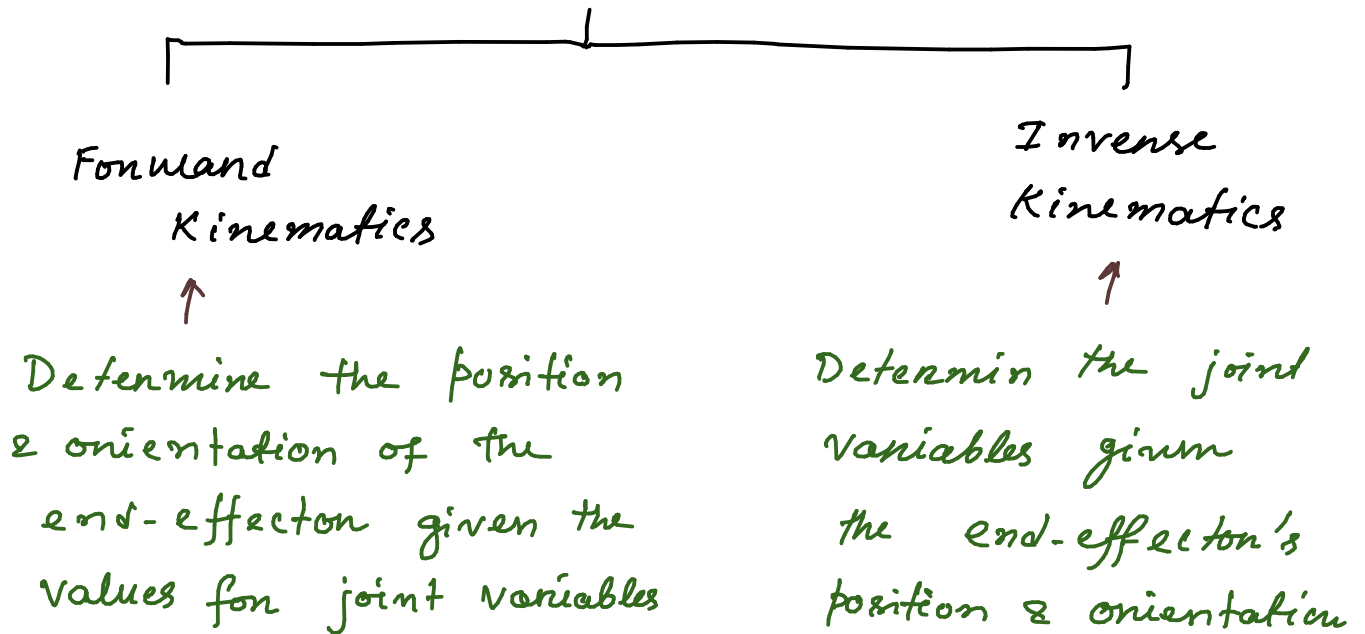


Part -1

Robot Kinematics ↙ Geometric description of a robot



- Recall: Revolute joint : allows rotational motion
- Prismatic joint : allows linear motion

* We consider that

- each joint of a manipulator allows only 1-DOF.

For:

- Ball & socket joint
(allows 2-DOFs)
 - Spherical wrist
(allows 3-DOFs)
- Will be represented as a succession of 1-DOF joints with links of zero length.

→ Some notations we follow:

- A manipulator with 'n' joints will have 'n+1' links.

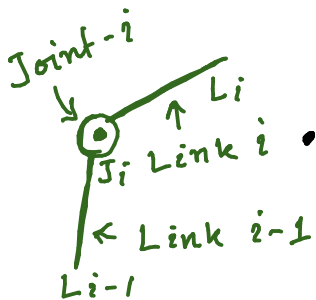
- Joints are numbered as

$$\{1, 2, 3 \dots n\}$$

Links are numbered as

$$\{0, 1, 2 \dots n\}$$

↑
assigned to the
Base



Joint-i connects link i-1 to link-i.

J_i L_{i-1} L_i

- When joint i is actuated, link-i moves

⇓

- Link - 0, which is Base, is always fixed when the joints are actuated.

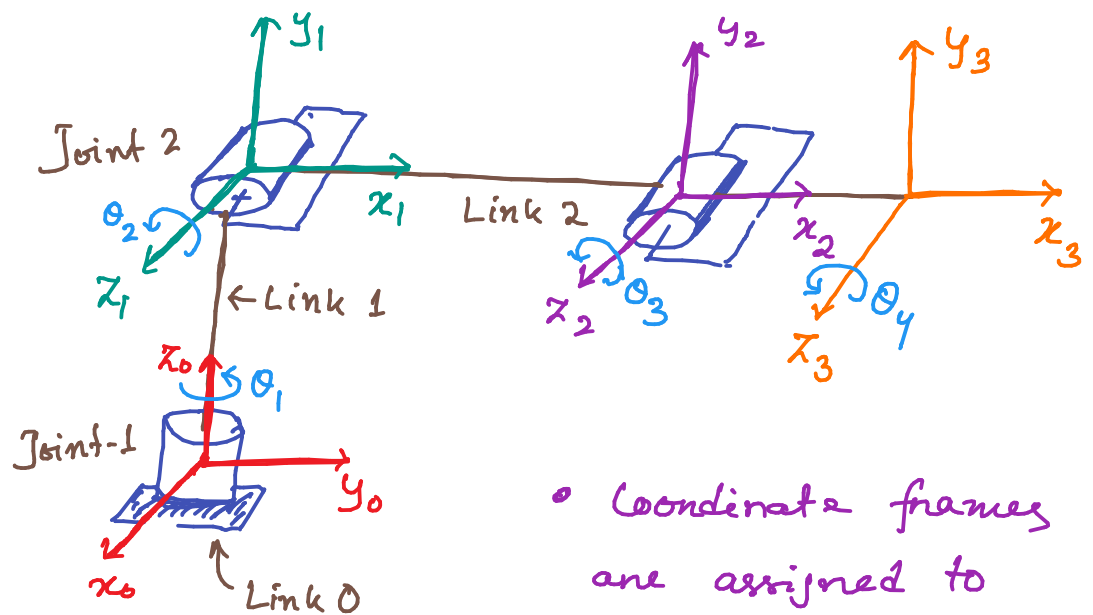
→ For i^{th} joint we associate a joint variable ' q_i ' where

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

θ_i : angle of rotation

d_i : amount of displacement.

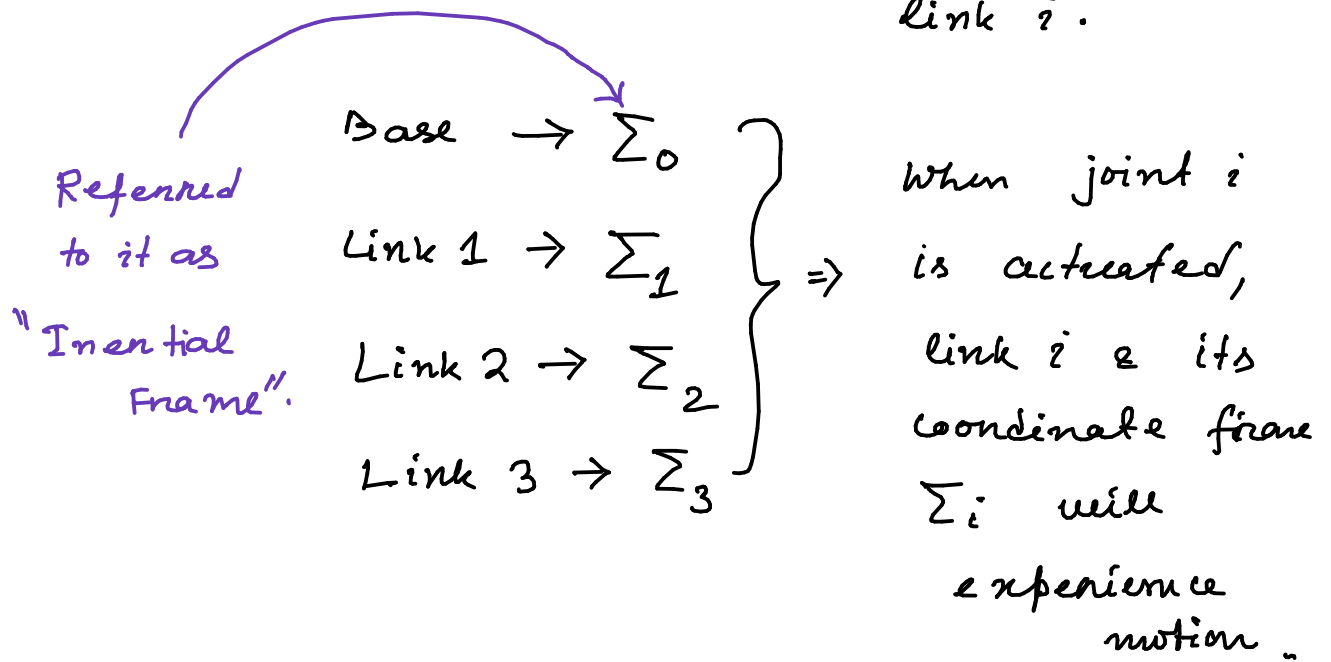
- To perform kinematic analysis we attach rigidly a coordinate frame to each link as in fig below.



- Coordinate frames are assigned to each link.

4-links $\Rightarrow$ 4 coordinate frames

- We attach coordinate frame Σ_i to link i .



- Let H_i be the homogeneous transformation matrix that gives position & orientation of Σ_i w.r.t. Σ_{i-1} .

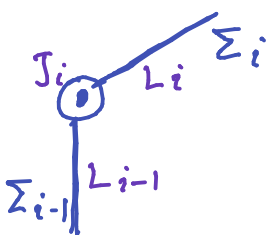
↓

The matrix H_i is not a constant matrix. It depends on the joint variable q_i (such as θ_i)

↓

H_i is represented as $H_i(q_i)$

↑ i th joint variable.



- By attaching the coordinate frame rigidly to the links of a manipulator, we assign

$\Sigma_0 \leftarrow$ Base coordinate frame

$\Sigma_n \leftarrow$ End-effector coordinate frame

$\Sigma_i \leftarrow$ i th link coordinate frame

- Denote the position & orientation of Σ_n w.r.t. Σ_0 by :

- the translation vector d_n^0
 - the rotation matrix R_n^0
- The corresponding homogeneous transformation matrix

$$H = \begin{bmatrix} R_n^0 & d_n^0 \\ 0 & 1 \end{bmatrix}$$

- By denoting the joint variables as $q_1, q_2, \dots, q_n$

we have :

$$H = H_1(q_1) H_2(q_2) \dots H_n(q_n)$$

where

$$H_i(q_i) = \begin{bmatrix} R_i^{i-1} & d_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

↓

$$H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} R_n^{n-1} & d_n^{n-1} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} R_n^0 & d_n^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_0^1 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} R_n^{n-1} & d_n^{n-1} \\ 0 & 1 \end{bmatrix} \quad \text{--- (*)}$$

↑

This transformation matrix (H)

helps to determine the

end-effector's configuration (posⁿ & orientation) given the joint variables θ_i .

↑

Forward kinematics.

→ A universal convention that is used to obtain the forward kinematics equation as in (*) is : Denavit-Hartenberg Convention.

- Denavit - Hartenberg (DH) Convention

↑ for open-chain manipulator

- A systematic procedure needs to be devised to define the relative position & orientation of two consecutive links : L_{i-1} & L_i

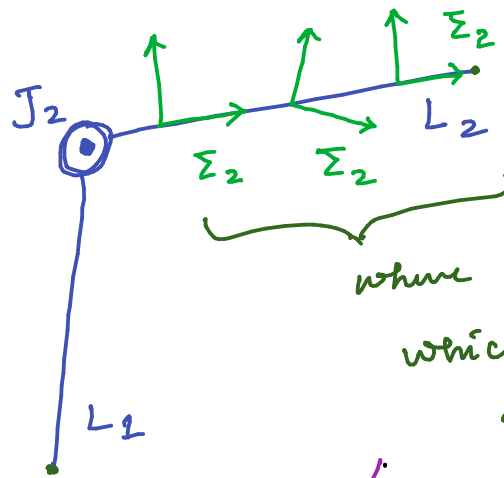
↓

For this, some rules need to be followed to assign the coordinate frames to the links.

- The problem appears due to the following reasons :

(1) where to place the origin of the coordinate frame in a particular link ?

- Which direction one should align the axes : x_i , y_i & z_i ?



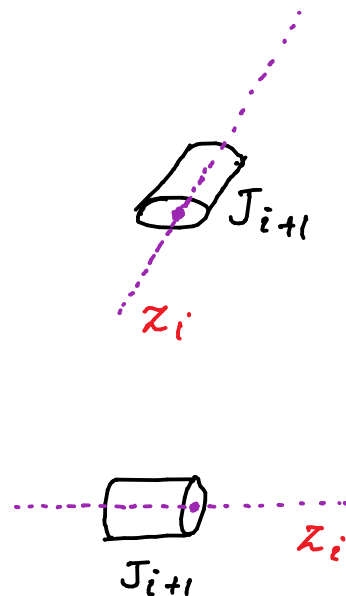
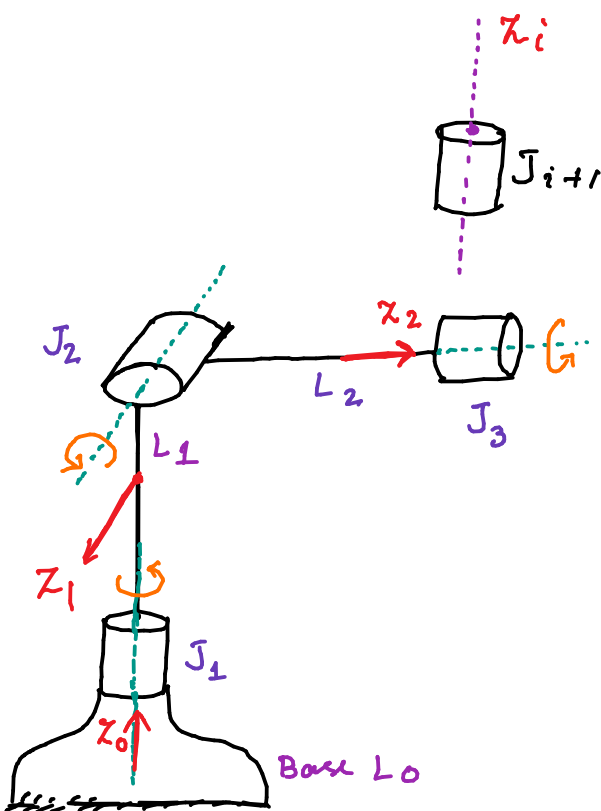
where to place & which orientation of Σ_2 should be ?

DH - Convention

• Step-1

choose axis z_i along the axis of J_{i+1} (joint $i+1$).

axis of rotation for revolute joint



- z_0 is the axis of actuation/rotation for J_1
- z_1 is the axis of rotation for J_2

- This axis convention comes from the fact that the actuation of J_i results in motion of L_i & its associated coordinate frame Σ_i .

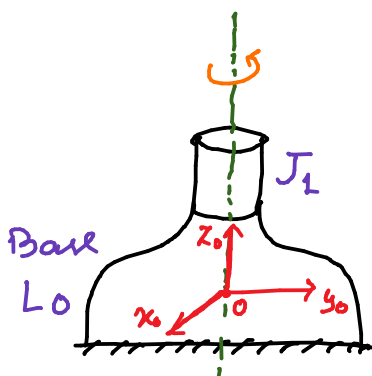
• Step-2

Once z_i -axes are fixed, we establish the base frame: Σ_0 . For this,

- Choose any point on z_0 as origin for Σ_0
- Choose then x_0 and y_0 axis orthogonal to each other, & they combinedly orthogonal to z_0 .

(all 3-axes are mutually orthogonal)

Choice of x_0 & y_0 are arbitrary as long as they are orthogonal)



↓
right-hand frame

(thumb → x -axis
index → y -axis
middle → z -axis)

Once Σ_0 is established, following steps are considered iteratively.

- Step-3

Following 3 - Cases are considered to establish Σ_i

(1) The axes z_{i-1} & z_i are not coplanar.

(2) The axes z_{i-1} & z_i intersect

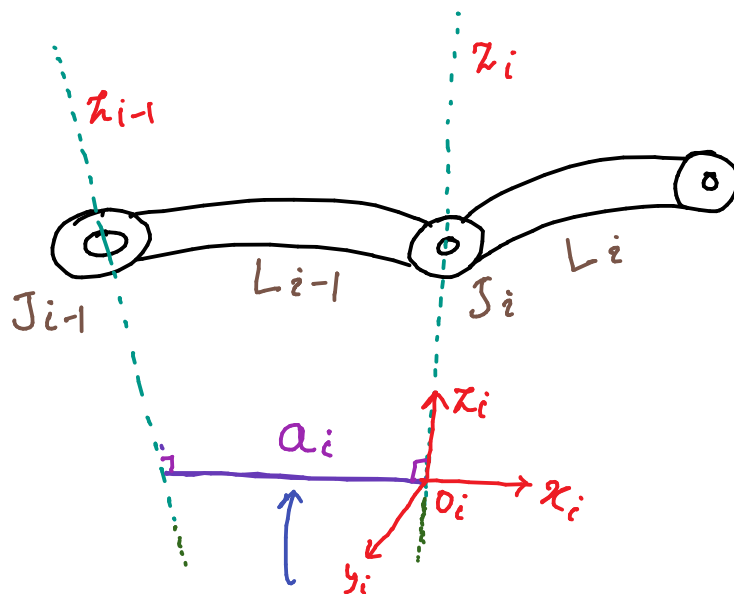
(3) The axes z_{i-1} & z_i are parallel.

the axes z_{i-1} & z_i are coplanar.

- Case 1 : z_{i-1} & z_i are not coplanar :

If z_{i-1} & z_i are not coplanar, then there exists a unique shortest line segment from z_{i-1} to z_i , perpendicular to both z_{i-1} & z_i . This line segment defines x_i & the point where it intersects z_i is the origin O_i .

Once x_i is fixed, then y_i is according to righted-frame rule.



Line segment joining z_{i-1} & z_i
 is perpendicular to both z_{i-1} & z_i
 Length of the
 Line segment is
 a_i .

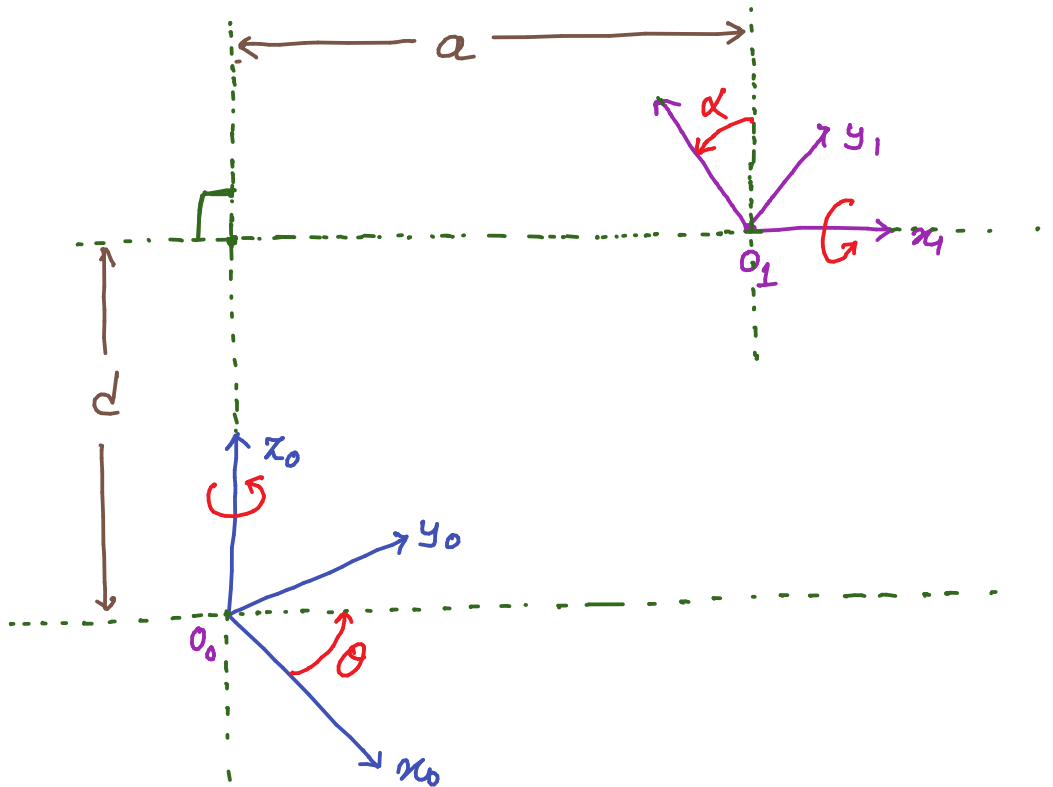
- The above procedure also satisfies some extra conditions as follows.

- ④ ... {
- axis x_i is perpendicular to z_{i-1}
 - axis x_i intersects z_{i-1} -axis.

Under these conditions there is
 some nice homogeneous transformation
 matrix H_i .



explained w.r.t coordinate frames Σ_0 & Σ_1 ,
 which satisfies the above conditions (④).



The conditions ① holds i.e.

- (i) axis - x_1 is perpendicular to x_0 -axis
- (ii) axis - x_1 intersects the axis x_0 .

Under these conditions $\exists$ unique numbers:

$\{a, d, \theta, \alpha\}$ such that

the homogeneous transformation matrix is

$$H = H_{x_0}^R H_{z_d}^T H_{x_a}^T H_{x_\alpha}^R$$

In DH-conversion, each HTM H_i is represented as a product of 4 above transformations.

$$H = H_{z0}^R H_{zd}^T H_{za}^T H_{xa}^R$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \\ \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha & a c_\theta \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha & a s_\theta \\ 0 & s_\alpha & c_\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \dots \Delta \Delta$$

$$= \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \leftarrow \begin{array}{l} \text{For the frames} \\ \Sigma_0 \text{ \& } \Sigma_1 \text{ we} \\ \text{have considered.} \end{array}$$

- In the previous figure, the condition: x_1 is perpendicular to x_0

$\Downarrow$

$$\boxed{x_1^T x_0 = 0} \dots \textcircled{1}$$

Since x_1 is in Σ_1 -frame, to perform the algebraic manipulation $\textcircled{1}$

... we need to represent x_1 w.r.t. to Σ_0 -frame

denote it as x_1^0

$$x_1^0 = R_1^0 x_1$$

$$R_1^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Note that

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ w.r.t. } \Sigma_1 \text{ frame}$$

↑
Since we do not know what are α_2, α_3 we represent R_1^0 as above.

$$\Rightarrow x_1^0 = R_1^0 x_1 = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

Now can perform the algebraic manipulation ① by replacing x_1 with x_1^0 .

i.e.

$$(x_1^0)^T x_0 = 0$$

$$\Rightarrow [r_{11} \ r_{21} \ r_{31}] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow r_{31} = 0$$

- Since R_1^0 is a rotation matrix, each row & column must be of unit (2-norm).

$\Downarrow$

$$r_{11}^2 + r_{21}^2 = 1 \quad (1^{st} \text{ col})$$

$$\& r_{32}^2 + r_{33}^2 = 1 \quad (\text{last row})$$

- Recall that every homogeneous transform¹ in DH convention can be represented as a product of 4 - basic transformations²

$$\{H_{x0}^R, H_{xd}^T, H_{xa}^T, H_{x\alpha}^R\}.$$

$\downarrow$

From this we have obtained in $\textcircled{\Delta\Delta}$ that

$$R_1^0 = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} = R_{x0} R_{x\alpha}$$

- Since $r_{11}^2 + r_{21}^2 = 1$, by comparing with R_1^0

We have
There exist unique θ & α s.t. these relations hold.

$$\begin{bmatrix} r_{11} \\ r_{21} \end{bmatrix} = \begin{bmatrix} c_\theta \\ s_\theta \end{bmatrix} \quad \& \quad \begin{bmatrix} r_{32} \\ r_{33} \end{bmatrix} = \begin{bmatrix} s_\alpha \\ c_\alpha \end{bmatrix}$$

Furthermore, by comparing:

$$\underbrace{\begin{bmatrix} c_0 & r_{12} & r_{13} \\ s_0 & r_{22} & r_{23} \\ 0 & s_\alpha & c_\alpha \end{bmatrix}}_{R_1^0} = \begin{bmatrix} c_0 & -s_0 c_\alpha & s_0 s_\alpha \\ s_0 & c_0 c_\alpha & -c_0 s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}$$

• After the evaluation of θ & α
 $\downarrow$

Since R_1^0 is a multiplicant of two rotational matrices, its other elements must be of following form:

$$\begin{aligned} r_{12} &= -s_0 c_\alpha & r_{13} &= s_0 s_\alpha \\ r_{22} &= c_0 c_\alpha & r_{23} &= -c_0 s_\alpha \end{aligned}$$

• For the HTM: $H = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$, we have

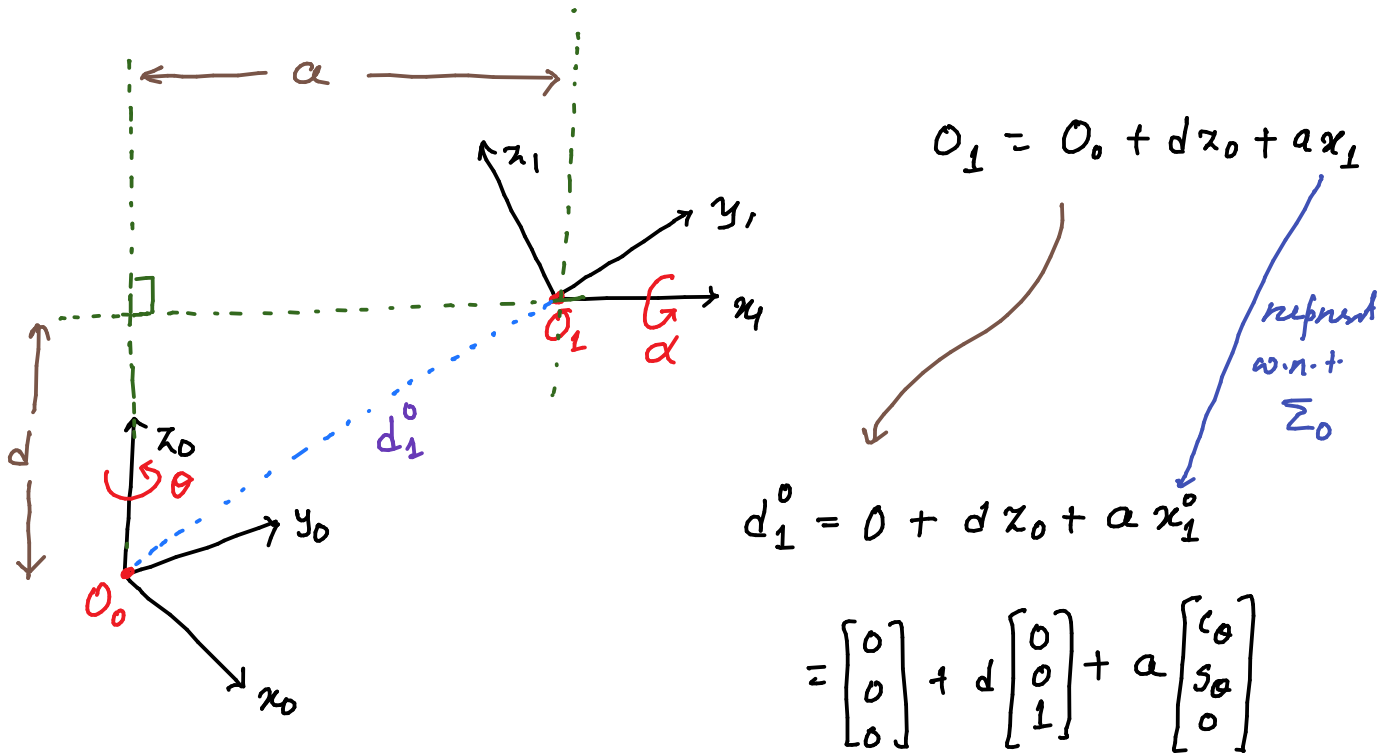
already computed θ & α using R_1^0 . Now
we compute a & d using d_1^0 .

→ Recall the 2nd assumption of DH-convention:

• x_1 -axis intersects z_0 -axis

$\Downarrow$

The displacement d_1^0 between the origins of Σ_0 & Σ_1 can be expressed as a linear combination of x_1 & z_0 .



$$\Rightarrow d_1^0 = \begin{bmatrix} a c_\theta \\ a s_\theta \\ d \end{bmatrix}$$

Comparing with the elements of d_1^0 , a & d can be determined.

Compare with the matrix in $\Delta \Delta$.

- The above discussion says that

Any homogeneous transformation, that satisfies two-stated DH-conditions, can be realized through 4-basis homogeneous transformations using the variables : $\{a, d, \theta, \alpha\}$.

→ Returning to the assignment of coordinate frames in DH - Conversion, we were discussing the case:

- Σ_{i-1} & Σ_i are not coplanar:

↳ We have discussed how to assign x_i -axis & the origin of Σ_i -frame.



With the above construction the stated two DH conditions:

- x_i is $\perp$ to Σ_{i-1} -axis
- x_i intersects Σ_i -axis

are satisfied.

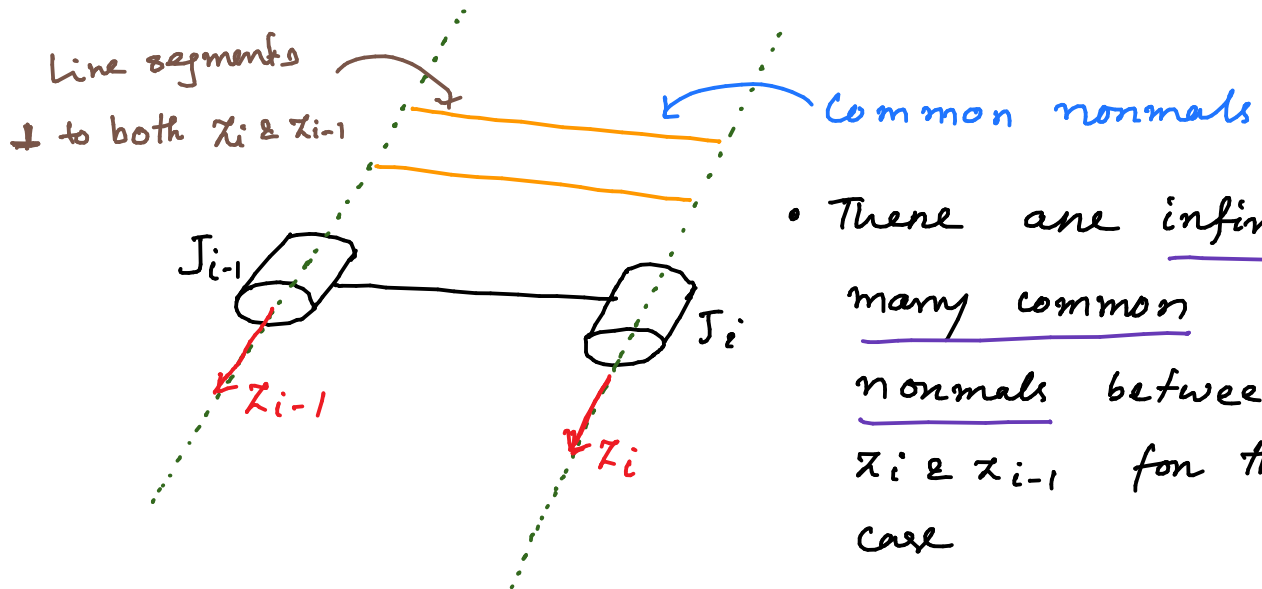


there exists a unique homogeneous transformation matrix of the following form between coordinate frames Σ_i & Σ_{i-1} :

$$H_i = H_{\Sigma_i O_i}^R H_{\Sigma_i d_i}^T H_{\Sigma_i a_i}^T H_{\Sigma_i \alpha_i}^R$$

→ The second case to set-up coordinate frame Σ_i :

- z_{i-1} is parallel to z_i



The DH first condition i.e.

z_i intersects z_{i-1} does not specify x_i completely (non-unique x_i 's).

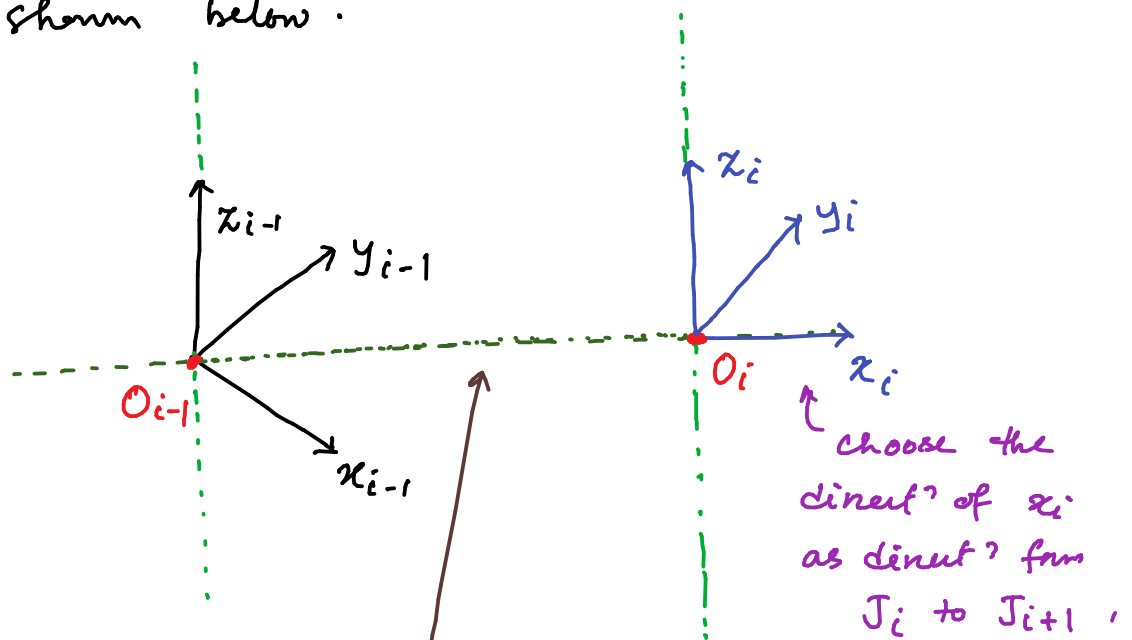


We are free to choose the origin of Σ_i at any point along z_i -axis.

- Since we are free to choose the origin O_i of Σ_i , we consider following approach:

- Out of infinitely many common normals (the line segment joining z_{i-1} & z_i axes & perpendicular to both), choose the common normal which passes through the origin O_{i-1} of Σ_{i-1} .

This line segment defines x_i as stated in previous case. This is shown below.



Note that in this case:

- $d_i = 0$
- $\alpha_i = 0$

Line segment joining z_i & z_{i-1} & perpendicular to both.

$\Downarrow$

x_i is along the directⁿ of this line segment.

Once x_i is fixed, the axis y_i is fixed according to right-hand frame rule.

→ In consideration of the 4-elements (a, d, θ, α) that determine the HTM H , we have following:

- $d_i = 0$ (since O_{i-1} & O_i are at same line)
- $\alpha_i = 0$ (since x_i & x_{i-1} are parallel & they are coplanar).

So we have only $\theta \neq 0$ & $a \neq 0$.

The HTM H needs to be computed accordingly for this case.

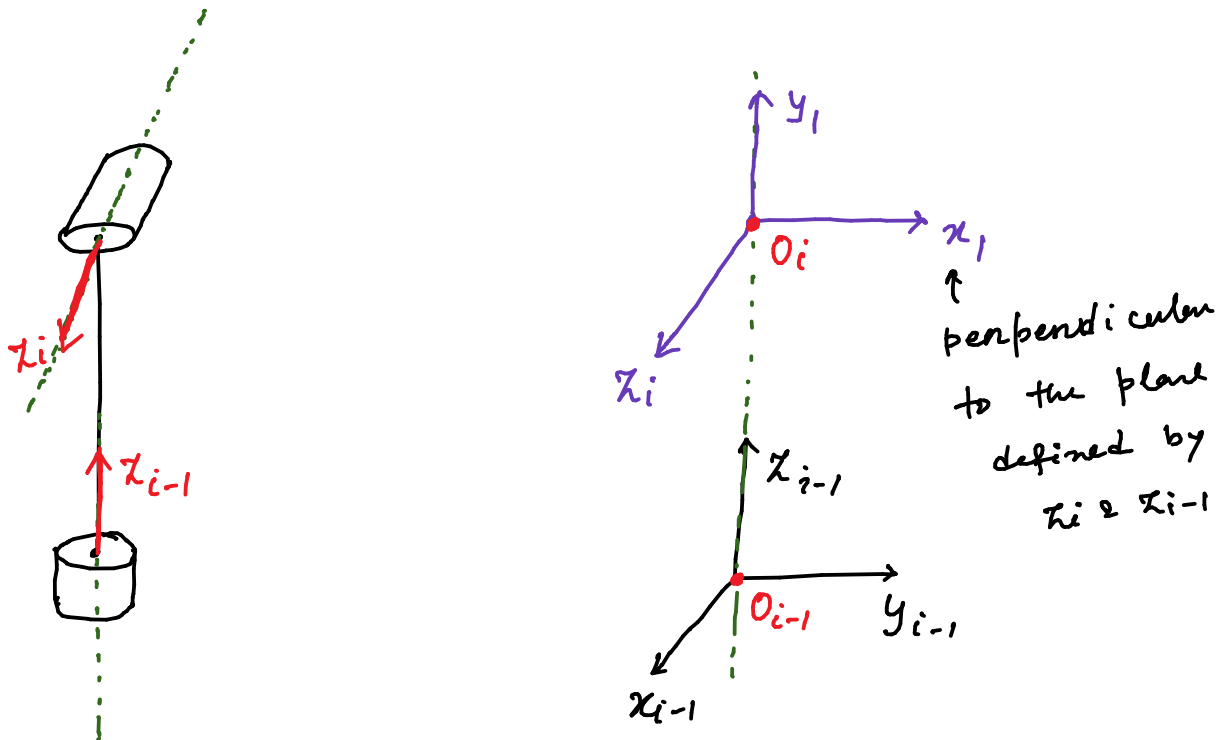
→ The 3rd case is:

- x_{i-1} -axis intersects x_i :

↑
 x_{i-1} & x_i are coplanar
↓

Choose x_i normal to the plane formed by x_{i-1} & x_i .

- In this case the origin O_i for Σ_i is at the intersection of z_{i-1} & z_i .



In this case the parameter

- $\alpha_i = 0$ since the displacement betⁿ O_{i-1} & O_i along x_i directⁿ is zero.

So in this case

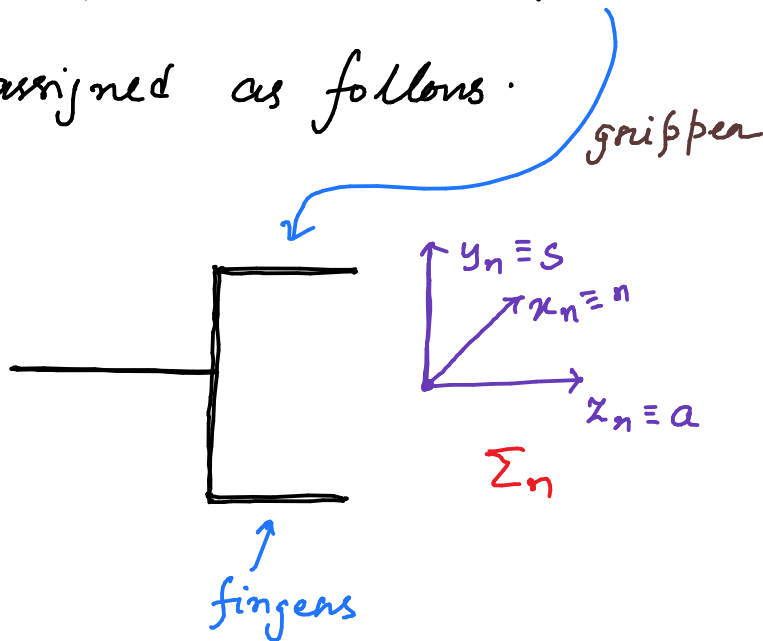
$$d_i \neq 0, \quad O_i \neq 0 \quad \& \quad \alpha_i \neq 0.$$

Based on these parameters, the homogeneous transformation matrix needs to be constructed.

- For an n -link manipulator, the previous procedure helps us to fix the coordinate frames for the links:

$$\{L_0, L_1 \dots L_{n-1}\}$$

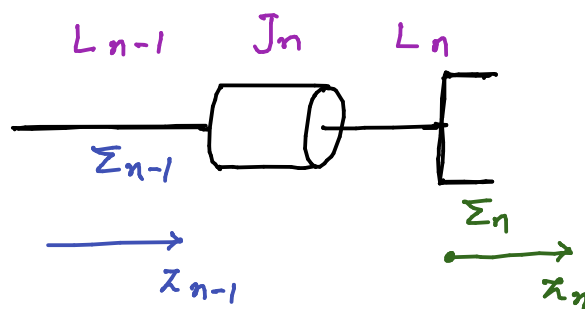
- The coordinate frame Σ_n for last link L_n , called "end-effector" will be assigned as follows.



- The origin O_n of Σ_n is placed symmetrically between the fingers of the gripper.
- The axes are labelled as follows:

- $z_n \equiv a \leftarrow$ gripper approaches to the object in this direct'
- $y_n \equiv s \leftarrow$ the direction along which the fingers slide to open & close
- $x_n \equiv n \leftarrow$ the direct' of normal to the plane formed by z_n & y_n .

→ Observe that the final joint motion is a rotation of the end-effector. In this case z_{n-1} & z_n coincide.



- In this case the transform' between Σ_{n-1} & Σ_n is
 1. translation along z_{n-1} by distance d_n ,
 2. followed or preceded by a rotation of θ_n about z_{n-1} .

• Some important facts:

1. Irrespective of the type of joints in the manipulator, the quantities

α_i & a_i are always constant for all i .

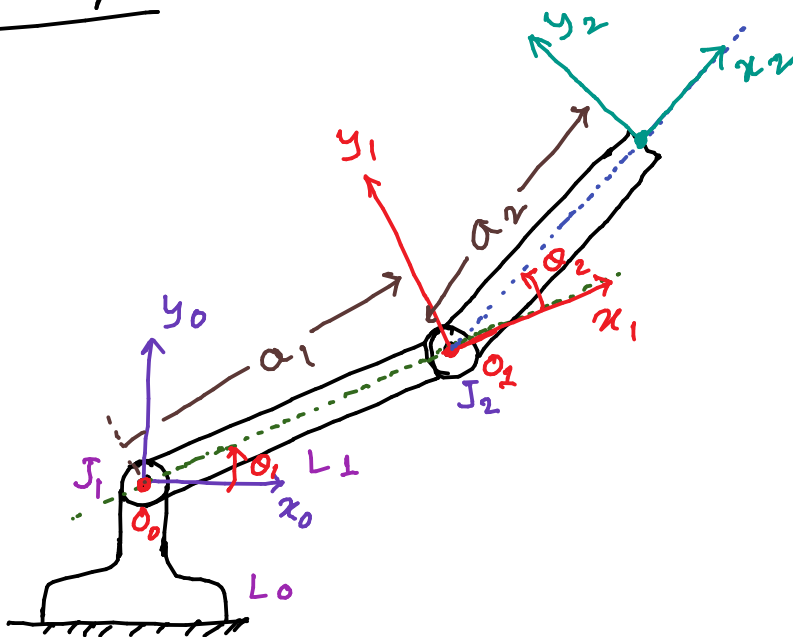
↑ characteristic of the manipulator.

2. If joint J_i is prismatic, then

θ_i is constant & d_i is variable.

3. If J_i is revolute, then d_i is constant & θ_i is variable.

→ Example



Three-link planar manipulator.

• The axis of rotation for both joints are normal to the plane.

Hence, z_0 -axis is normal to the plane.

The origin O_0 for Σ_0 needs to be chosen on the z_0 -axis. Hence, we have chosen O_0 at the intersection of z_0 -axis & the plane, i.e. the center of J_1 . The dirct' of z_0 & y_0 are arbitrarily assigned.

$\Downarrow$

we have fixed Σ_0 .

- Note that z_0 & z_1 axes are parallel.

$\Downarrow$

there are infinitely many common normals.

$\Downarrow$

According to DH Convention we choose the common normal which passes through origin O_0 of Σ_0 . This common normal on the line segment defines the dirct' of z_1 .

Then y_1 is according to righthand-frame rule.

$\Downarrow$

Σ_1 is established

$\swarrow$
Recall, in this
case: $d_1 = d = 0$
 $\alpha_1 = 0$ }

- Σ_2 - axis is at the tip of link-3.

Note that Σ_1 & Σ_2 axes are also parallel. Hence, Σ_2 is also fixed according to the previous discussion. & it is shown in the figure.

↓
 In this case $\left. \begin{array}{l} d_2 = d = 0 \\ \alpha_2 = 0 \end{array} \right\}$

Coordinate frames (Links)	a_i	α_i	d_i	θ_i
$\Sigma_1 (L_1)$	a_1	0	0	θ_1
$\Sigma_2 (L_2)$	a_2	0	0	θ_2

Hence the HTMs are:

$$H_i = H_{\Sigma_i \theta_i}^R H_{\Sigma_i d_i}^T H_{\Sigma_i a_i}^T H_{\Sigma_i \alpha_i}^R \quad \text{for } i = 1, 2$$

$$H_1 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & a_1 c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & a_1 s_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & a_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & a_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1 H_2$$