

ELL 406
ROBOT KINEMATICS

Part - 2

→ INVERSE KINEMATICS

↑ Given the end-effector's configuration,
compute joint variables of a
manipulation.

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Finding of joint variables in terms
of the end-effector's position &
orientation.

- Recall that the joint variables are q_i for
 i^{th} joint. The homogeneous transformation
matrix

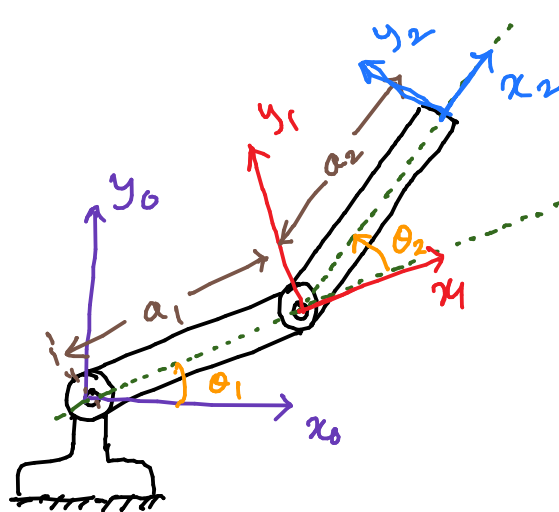
$$H = H_1 H_2 H_3 \cdots H_n$$

where $H_i = \begin{bmatrix} R_i^{i-1} & d_i^{i-1} \\ 0 & 1 \end{bmatrix}$

↓

$$H_i(q_i)$$

→ Consider the 2-link planar manipulator.



Recall that we had obtained the homogeneous transformation matrix as:

$$H = H_1 H_2$$

$$\theta := \theta_1 + \theta_2$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & a_1 c_\theta + a_2 c_\theta \\ s_\theta & c_\theta & 0 & a_1 s_\theta + a_2 s_\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Assume that for a given configuration of the end-effector, the HTM is given as follows:

$$H = \begin{bmatrix} 0 & -1 & 0 & 0.5 \\ 1 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the joint variables θ_1 & θ_2 we need to solve the following set of nonlinear trigonometric equation:

Solve them to obtain θ_1 & θ_2

$$\left\{ \begin{array}{l} c_\theta = 0 \quad ; \quad -s_\theta = -1 \\ a_1 c_\theta + a_2 c_\theta = 0.5 \\ s_\theta = 1 \\ a_1 s_\theta + a_2 s_\theta = -1.5 \end{array} \right.$$

↑ inverse kinematics problem.

- In the above 2-link manipulator:

$$\begin{bmatrix} x_2^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} a_1 c_\theta + a_2 c_\theta \\ a_1 s_\theta + a_2 s_\theta \end{bmatrix}$$

We know the following quantities:

- x_2^0 : End-effector's x -coordinate w.r.t. base frame Σ_0
- y_2^0 : End-effector's y -coordinate w.r.t. Σ_0
- a_1 : length of link-1
- a_2 : length of link-2

→ The unknown quantities are

- joint variables θ_1 & θ_2
- In an inverse kinematics problem, we need to solve simultaneously 12 non-linear trigonometric equations to obtain the joint variables q_i .

↑ Solving this is a difficult problem!

To solve the equations

Analytic Method

Numerical
Iterative method

→ In the analytical method, with some specific assumption, one can solve the problem geometrically by performing
"Kinematic Decomposition".

→ Kinematic Decoupling

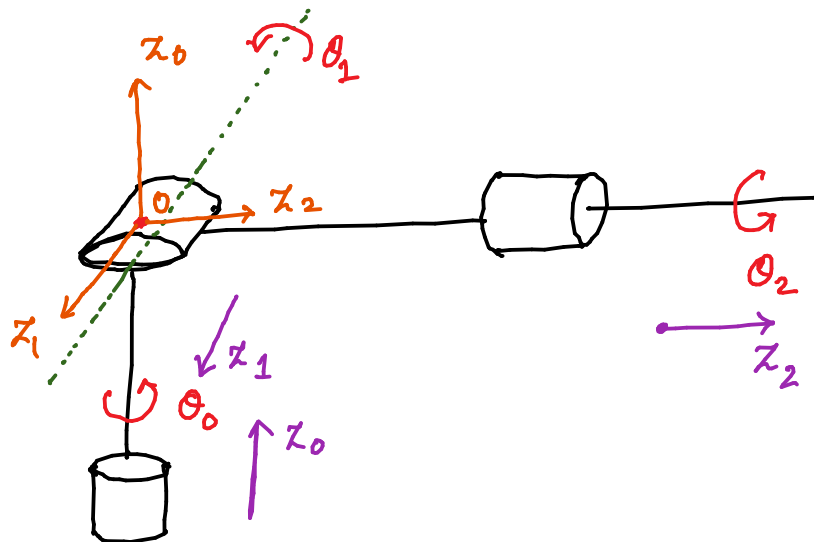
- Assumption:

The manipulator has 6-joints & the last 3-joint axes intersect at a point.

↓ With this assumption

Inverse kinematic problem can be decoupled into two following simpler problems:

1. Inverse Position kinematics
2. Inverse orientation kinematics

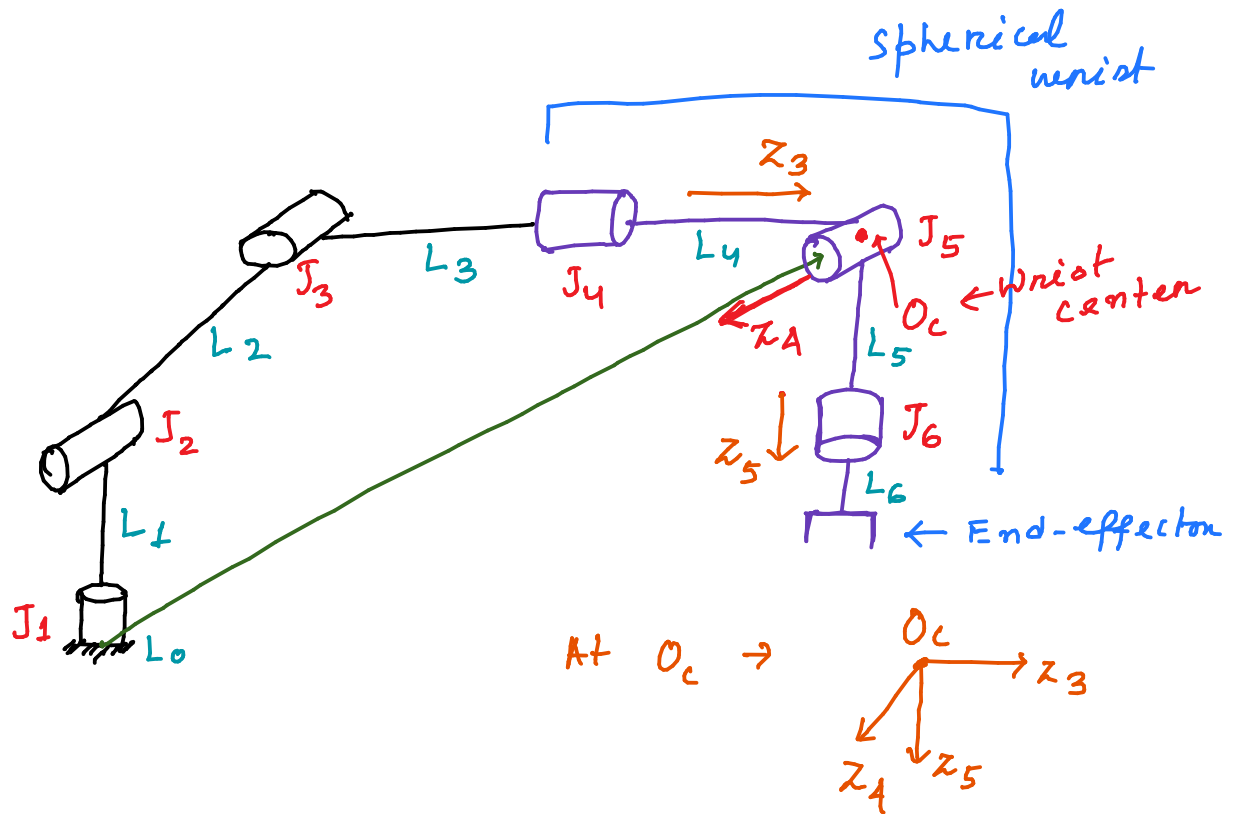


For a manipulator, having spherical wrist, the joint axes z_0 , z_1 , & z_2 intersect at O , satisfying the assumption.

- For a 6-DOF manipulator with a spherical wrist the inverse kinematics problem can be separated into following two simpler problems:

1. First, find the position of the intersection of the wrist axes (inverse position kinematics), then
The intersection point is called "wrist center"
2. second, find the orientation of the wrist (inverse orientation kinematics)

→ A 6-DOF spherical wrist manipulator:

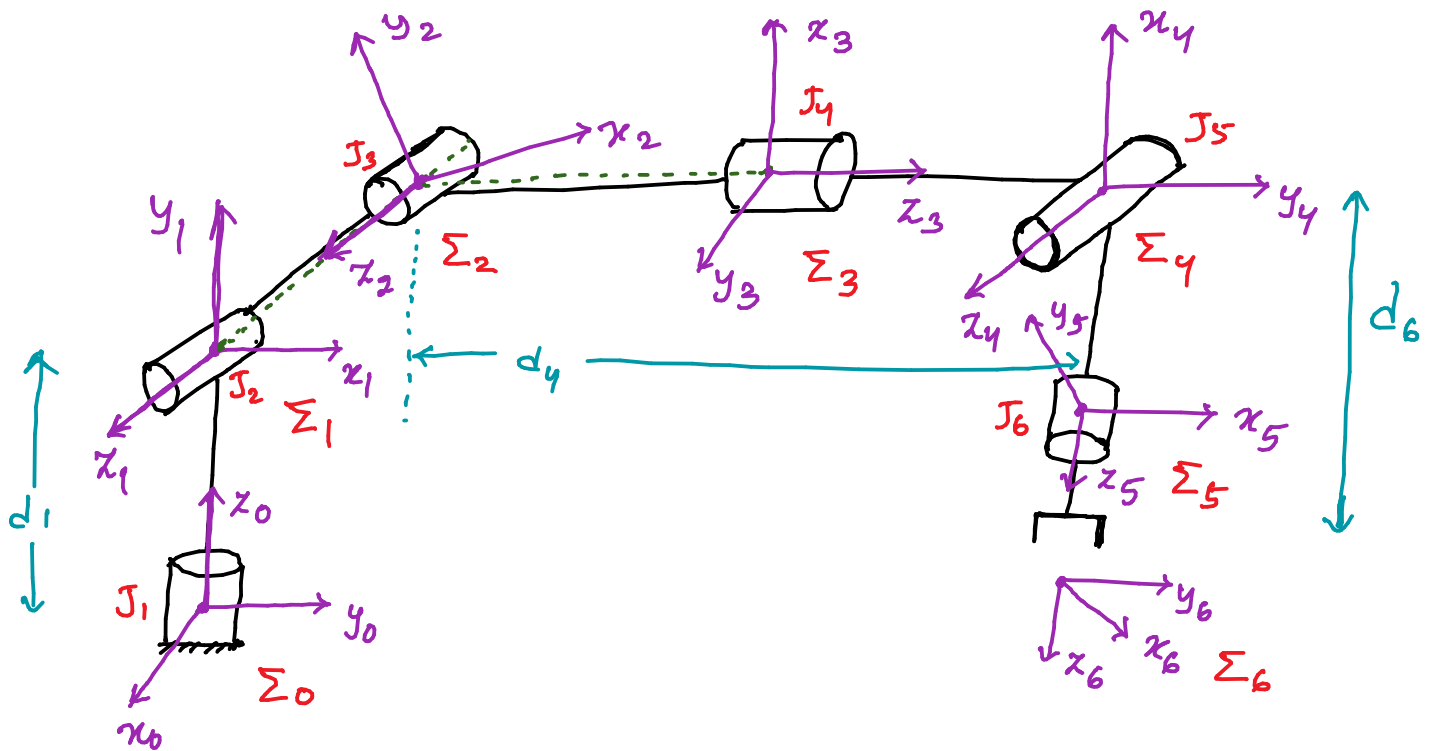


- Inverse position problem can be addressed using geometric approach (simple geometry & trigonometric manipulation).
- Inverse Orientation problem can be addressed using Euler-angle parameterization (with Z-Y-Z Euler angle transformation matrices).

* Refer to "Robot Modeling & Control"
by M.W. Spong, S. Hutchinson &
M. Vidyasagar
book (Section 3.3) for
details)

- Another approach for addressing inverse kinematics problem is through "numerical iterative" method. Here, one may use "Newton-Raphson" method to solve numerically a set of non-linear algebraic equations.

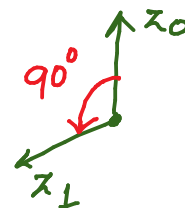
→ DH - table construction for 6-DOF spherical link manipulator:



1. z_0 & x_1 intersect:

- $a_1 = 0$
- $\alpha_1 = 90^\circ$

(z_0 needs to be rotated counterclockwise about x_1 to reach x_1)



- $d_1 \neq 0$ (displacement of O_0 & O_1 along z_0 direction)
- $\theta_1 \neq 0$

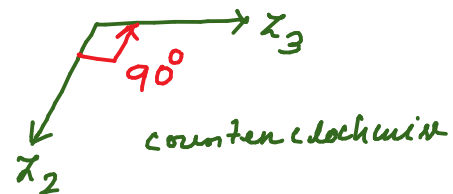
2. π_1 & π_2 are parallel (planar)

- π_2 axis is along the direction of normal to π_1 & π_2 .
- $d_2 = 0$
- $\alpha_2 = 0$
- $a_2 \neq 0$ (displacement between O_1 & O_2 along π_2 direction)
- $\theta_2 \neq 0$

3. π_2 & π_3 intersects

- π_3 -axis should be normal to the plane spanned by π_2 & π_3
- $a_3 = 0$ (No displacement bet' O_2 & O_3 along π_3 -direction)
- $\alpha_3 = 90^\circ$

(π_2 needs to be rotated 90° about π_3 -axis to reach π_3)



- $\theta_3 \neq 0$

- $d_3 = 0$, since J_3 is a revolute joint, there can not be displacement along x_2 - direct.

$\uparrow$
 d_3 is the displacement
 between O_2 & O_3 along x_2
 direction.

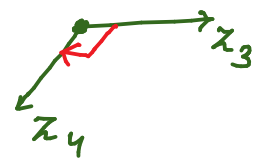


we need to place O_3 at O_2 . This
 is possible since O_3 can be placed
 anywhere along x_3 .

[* For Σ_0 & Σ_1 we do not need to set $d_1 = 0$,
 since d_1 is the displacement betⁿ O_0 & O_1
 along x_0 directⁿ. This is possible by considering
 the physical length of link L_1 as d_1 .]

4. x_3 & x_4 intersects:

- x_4 is perpendicular
to x_4 & x_3 plane.

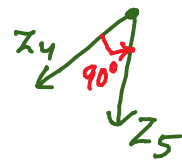


- $\alpha_4 = -90^\circ$ (x_3 has to rotate 90° in
clockwise direction about x_4
to reach x_4)
- $a_4 = 0$ (No displacement along x_4
directⁿ betⁿ O_3 & O_4)

- $d_4 \neq 0$, since the displacement between O_3 & O_4 is possible along z_3 direction, which is $d_4 = L_3 + L_4$ (since O_3 is placed at O_2).

- $O_4 \neq 0$

5. z_4 & z_5 intersects



- z_5 is normal to the plane z_4 & z_5 .
- $\alpha_5 = 90^\circ$ (z_4 has to be rotated 90° in counter clockwise about z_5 to reach z_5)
- $a_5 = 0$
- $d_5 = 0$, since the displacement between

O_4 & O_5 is not possible along z_4 -direction.



O_5 needs to be placed at O_4 , which lies on the z_5 -axis.

- $O_5 \neq 0$

6. z_5 & z_6 are along same direct?:

- $d_6 \neq 0$ ($d_6 = L_5 + L_6$, since O_5 is placed at O_4)
- $\alpha_6 = 0$
- $a_6 = 0$
- $\theta_6 \neq 0$

→ DH table for 6-DOF spherical link manipulator:

Link	a_i	α_i	d_i	θ_i
L_1	0	90°	d_1	θ_1
L_2	a_2	0	0	θ_2
L_3	0	90°	0	θ_3
L_4	0	-90°	d_4	θ_4
L_5	0	90°	0	θ_5
L_6	0	0	d_6	θ_6

- Recall the figure:

