

ELL406
ROBOT KINEMATICS

Part-3

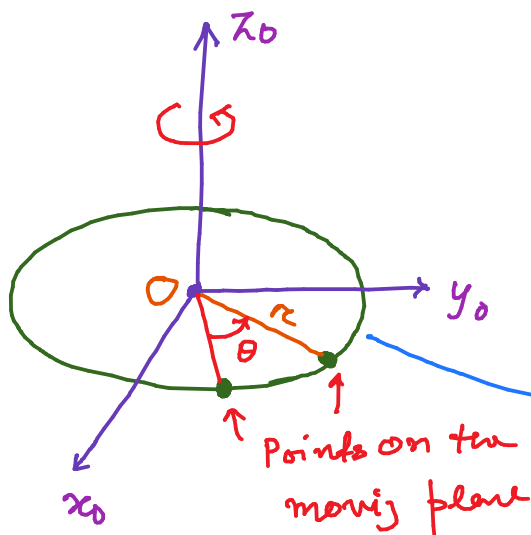
→ VELOCITY KINEMATICS

Forward velocity kinematics → { Given the joint velocities, determine the end-effector's velocity.

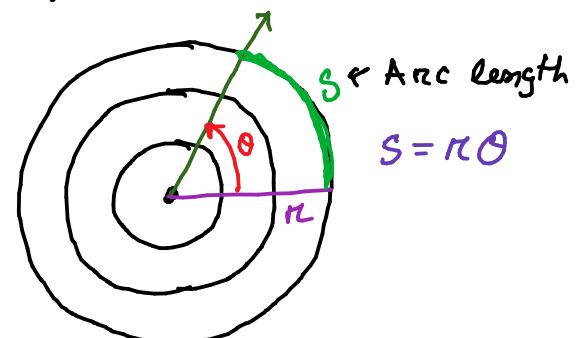
Inverse velocity kinematics → { Given the end-effector's velocity, determine the joint velocities.

• Angular velocity (Fixed axis case) :

When a rigid body moves in a pure rotation about a fixed axis, every point in the rotating plane moves in a circle.



Center of the circle O is on the axis of rotation.

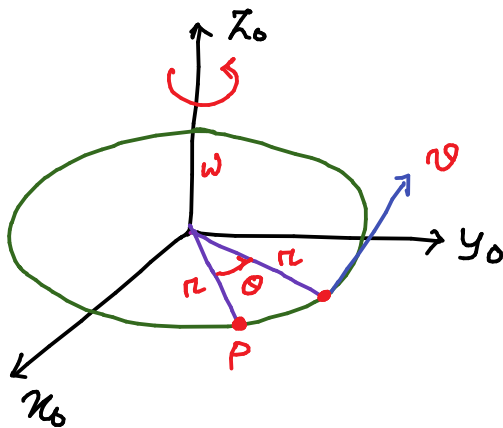
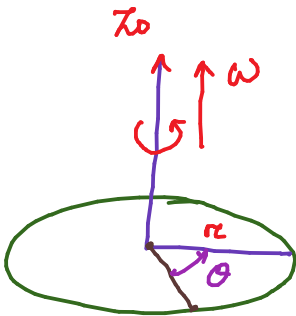


- Arc length : $s = r\theta$

Let u be a unit vector along the direction of axis of rotation. Then,

- angular velocity $\omega = \frac{d\theta}{dt} u$

↑ the rate of change of angular position (angle θ) w.r.t. time t . Its direction is along the direction of rotation of the axis (here z_0).



- Any point 'p' in the plane x_0-y_0 has also a linear velocity v when there is motion in x_0-y_0 plane.

- The relation between angular velocity ω & linear velocity v is:

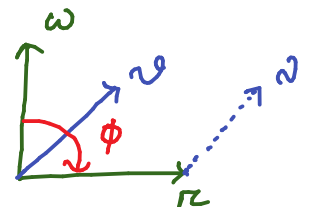
$$v \Delta t = \Delta s = r \Delta \theta$$

$$\Rightarrow v = r \frac{\Delta \theta}{\Delta t}$$

ω

$$v = \omega \times r$$

by providing appropriate directions vector cross product



- Vector cross-product:

Let i, j & k be three unit vectors as

follows:

$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using these unit vectors, any vector

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be represented as:

$$x = x_1 i + x_2 j + x_3 k$$

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then the cross-product

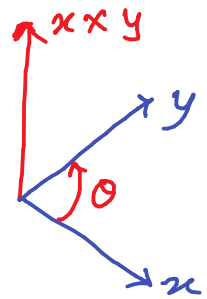
of x & y is:

$$\bullet \quad x \times y = \det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$= (x_2 y_3 - x_3 y_2) i + (x_3 y_1 - x_1 y_3) j + (x_1 y_2 - x_2 y_1) k$$

- The magnitude of $x \times y$ is:

$$\|x \times y\| = \|x\| \|y\| |\sin \theta|$$



where θ is the angle between x & y whose direction is determined by right hand rule.

- To study velocity kinematics, we need some basics on "Skew Symmetric Matrices".

→ Skew Symmetric Matrices:

↖ set of such matrices " $so(3)$ "

A $n \times n$ matrix S is said to be skew symmetric if

$$S^T + S = 0 \quad \dots (*)$$

Let
$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

Then $(*)$ gives rise to following 9 equations:

$$s_{ij} + s_{ji} = 0 \quad \text{for } i, j = 1, 2, 3$$

... $(**)$

- The equation $S_{ij} + S_{ji} = 0$

$\Downarrow$

$$S_{ii} = 0$$

$\Downarrow$

{ The diagonal entries of S are 0
 & the off-diagonal entries satisfy: $S_{ij} = -S_{ji}$

$\Downarrow$

- S contains only three independent variables: s_1, s_2 & s_3 & we have:

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

- If $a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ is a vector, we define

the skew symmetric matrix associated with 'a' as follows:

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

For instance :

$$S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{for } i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S(j) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{for } j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

→ Properties of Skew Symmetric Matrices :
(SSM)

- The set $\mathfrak{so}(3)$ is a Lie-algebra (vector space with a suitably defined product operation).

1. The operator S is linear, i.e.

$$S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$$

for any vectors $a, b \in \mathbb{R}^3$ & scalars $\alpha, \beta \in \mathbb{R}$.

2. For any vectors $a \in \mathbb{R}^3$ & $b \in \mathbb{R}^3$:

$$S(a)b = a \times b$$

cross-product

3. Let R be a rotation matrix $\& a \in \mathbb{R}^3$

$R \in SO(3)$ $\& a \in \mathbb{R}^3$. Then,

$$R S(a) R^T = S(Ra)$$

Proof

One can verify that for every vector $a, b \in \mathbb{R}^3$ $\& R \in SO(3)$, we have:

$$R \cdot (a \times b) = Ra \times Rb$$

$$\begin{aligned} R S(a) \underbrace{R^T b}_p &= R S(a) p \\ &= R (a \times p) \\ &= (Ra) \times (Rp) \\ &= (Ra) \times (R R^T b) \\ &= (Ra) \times b \\ &= S(Ra) \times b, \text{ since } S(a) p = a \times p \end{aligned}$$

$$\Rightarrow \underbrace{R S(a) R^T}_{\uparrow} = S(Ra)$$

Similarity transformation of $S(a)$

4. For an $n \times n$ SSM: S & any vector $k \in \mathbb{R}^n$

$$k^T S k = 0$$

show it

→ The derivative of a Rotation matrix :

let a rotation matrix R be a function of variable θ (for instance joint variable).

↓

Hence $R = R(\theta) \in SO(3)$ for all θ

⇓

$$R(\theta) R(\theta)^T = I$$

Differentiating both sides of above w.r.t θ

$$\left[\frac{d}{d\theta} R(\theta) \right] R(\theta)^T + R(\theta) \left[\frac{d}{d\theta} R(\theta)^T \right] = 0 \quad \dots \textcircled{1}$$

By defining a matrix :

$$S := \left[\frac{d}{d\theta} R(\theta) \right] R(\theta)^T$$

↓

$$S^T = R(\theta) \left[\frac{d}{d\theta} R(\theta)^T \right]$$

Then equation 1 implies

$$S + S^T = 0$$

$\Downarrow$

The matrix $S = \left[\frac{d}{d\theta} R(\theta) \right] R(\theta)^T$

is a skew symmetric matrix.

Ex :

Let $R(\theta) = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (about z -axis)
rot

$$\frac{d}{d\theta} R(\theta) = \begin{bmatrix} -s_\theta & -c_\theta & 0 \\ c_\theta & -s_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{d}{d\theta} R(\theta) \right] R(\theta)^T = \begin{bmatrix} -s_\theta & -c_\theta & 0 \\ c_\theta & -s_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$
 $R(\theta) = R_z(\theta)$

$$= \begin{bmatrix} -s_\theta c_\theta + c_\theta s_\theta & -s_\theta^2 - c_\theta^2 & 0 \\ c_\theta^2 + s_\theta^2 & c_\theta s_\theta - s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S(k)$$

$$k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From the previous example :

$$\left[\frac{d}{d\theta} R_x(\theta) \right] R_x(\theta)^T = S(k)$$

$$\Rightarrow \left[\frac{d}{d\theta} R_x(\theta) \right] = S(k) R_x(\theta)$$

Similarly

$$\left[\frac{d}{d\theta} R_x(\theta) \right] = S(i) R_x(\theta)$$

$$\left[\frac{d}{d\theta} R_y(\theta) \right] = S(j) R_y(\theta)$$

← Show them.

→ Assume that a rotation matrix is
time-varying i.e. $R = R(t) \in SO(3), \forall t \geq 0$.

↓

It can be considered that
the joint variable θ

is time varying i.e. $\theta = \theta(t)$

$$\Rightarrow R(t) = R(\theta(t))$$

- Let $R(t)$ be continuously differentiable
functⁿ of time t .

Then taking derivative of $R(t)$ w.r.t.
time t :

$$\frac{d}{dt}[R(\theta(t))] = \frac{dR}{d\theta} \cdot \frac{d\theta}{dt} \\ = [SR(\theta)] \dot{\theta}$$

Already shown
then $\frac{dR}{d\theta} = SR(\theta)$

• Let R be $\rightarrow R_x(t) = R_x(\theta(t))$

Then

$$\frac{dR_x}{dt} = [S(i)R_x(t)] \dot{\theta}, \quad i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\dot{\theta}$ is a scalar variable. Since S is
a linear operation, we have

$$\dot{\theta} S(i) = S(i\dot{\theta}) = S(\omega(t))$$

Hence

Angular velocity

$$\frac{dR_x}{dt} = S(\omega(t))R_x(t)$$

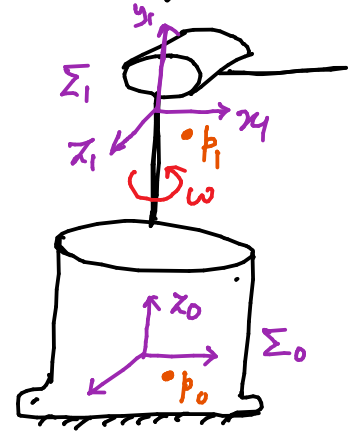
$\omega = \dot{\theta} u$
↑
unit vector
which is i here.

is the angular velocity
of the rotating frame w.r.t.
the fixed frame at time t .

Consider a point p rigidly attached to a moving coordinate frame. Then the coordinates of p w.r.t. the fixed frame is :

$$p_0 = R_1^0 p_1$$

R_1^0 is the rotation matrix of Σ_1 w.r.t. Σ_0 .



Note that R_1^0 is time varying $R_1^0(t)$ since Σ_1 is a moving frame w.r.t. time.

$\Downarrow$

The point p is also time-varying.

$\Downarrow$

The coordinates p_0 of p w.r.t. Σ_0 is time-varying, whereas the coordinates p_1 of p w.r.t. Σ_1 is not time-varying, since it is rigidly attached to Σ_1 .

$\Downarrow$

$$\frac{d}{dt} p_0(t) = \frac{d}{dt} [R_1^0(t) p_1]$$

$$\begin{aligned}
\frac{d}{dt} p_0(t) &= \frac{d}{dt} [R_1^0(t) p_1] \\
&\quad \text{Linear velocity} \nearrow \\
&= \frac{d}{dt} [R_1^0(t)] p_1 \\
&= S(\omega(t)) \underbrace{R_1^0(t) p_1}_{p_0(t)} \\
&= S(\omega) p_0 \\
&= \omega \times p_0 \quad \left(\text{Using the rule } S(a)p = a \times p \right)
\end{aligned}$$

Result :

$$\boxed{\frac{d R_1^0(t)}{dt} = S(\omega(t)) R_1^0(t)} \dots \checkmark$$

↓

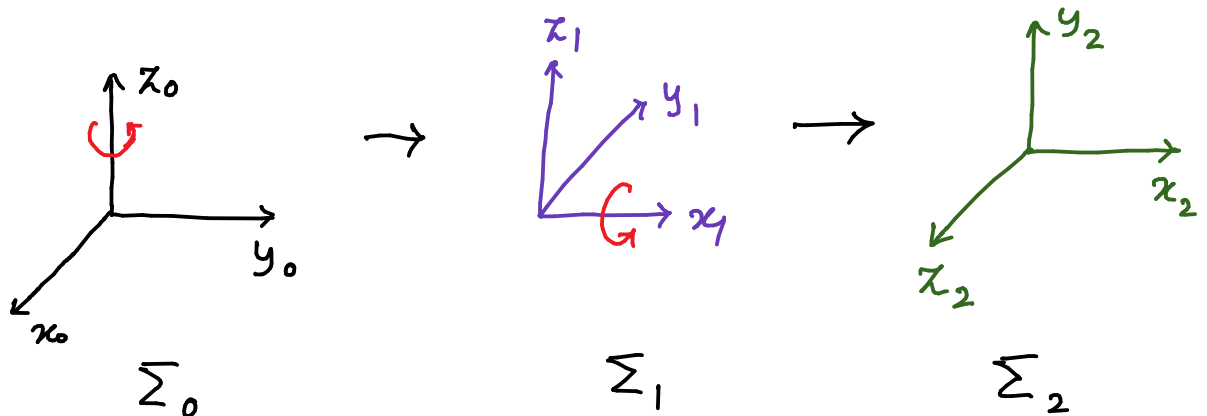
This says that if the instantaneous orientation of a coordinate frame Σ_1 w.r.t. coordinate frame Σ_0 is given by R_1^0 , then the angular velocity of Σ_1 is directly related to the derivative of R_1^0 through eqⁿ $\checkmark$.

→ Resultant Angular velocity due to the rotation of several coordinate frames:

- Composition of angular velocities of two moving frames: Σ_1 & Σ_2 w.r.t. a fixed frame Σ_0 .



Recall the composition of rotations (current frame) .



Let R_1^0 : be the rotation matrix representing the relative orientation of Σ_1 w.r.t. Σ_0

R_2^1 : be the rotation matrix representing the relative orientation of Σ_2 w.r.t. Σ_1 .

Then, the rotation matrix representing the relative orientation of Σ_2 w.r.t. Σ_0 is

$$\mathcal{R}_2^0 = \mathcal{R}_1^0 \mathcal{R}_2^1$$

- Now assume that $\mathcal{R}_1^0$ & $\mathcal{R}_2^1$ are both time-varying. Then, the above relatⁿ can be written as:

$$\mathcal{R}_2^0(t) = \mathcal{R}_1^0(t) \mathcal{R}_2^1(t)$$

Notations:

$$\frac{d}{dt} \mathcal{R}_1^0(t) \rightarrow \dot{\mathcal{R}}_1^0$$

Recall that $\dot{\mathcal{R}}_1^0 = S(\omega) \mathcal{R}_1^0$

- ω_{ij} : angular velocity vector that corresponds to the derivative of $\mathcal{R}_j^i$
- ω_{ij}^k : angular velocity vector corresponding to the derivative of $\mathcal{R}_j^i$ & expressed relative to coordinate frame Σ_k .

→ We have the composition rule

$$\mathcal{R}_2^0(t) = \mathcal{R}_1^0(t) \mathcal{R}_2^1(t)$$

$$\Rightarrow \boxed{\dot{\mathcal{R}}_2^0 = \dot{\mathcal{R}}_1^0 \mathcal{R}_2^1 + \mathcal{R}_1^0 \dot{\mathcal{R}}_2^1} \dots \textcircled{1} \quad \begin{array}{l} \text{(taking derivative} \\ \text{on both sides)} \end{array}$$

Using the previous relatⁿ:

$$\boxed{\dot{\mathcal{R}}_2^0 = S(\omega_{02}^0) \mathcal{R}_2^0} \dots \textcircled{2}$$

the resultant angular velocity experienced by frame Σ_2 w.r.t. fixed frame Σ_0 .

Similarly, the term:

$$\boxed{\dot{\mathcal{R}}_1^0 \mathcal{R}_2^1 = S(\omega_{01}^0) \mathcal{R}_1^0 \mathcal{R}_2^1 = S(\omega_{01}^0) \mathcal{R}_2^0} \dots \textcircled{3}$$

The angular velocity of Σ_1 that results from changing of $\mathcal{R}_1^0$ (time derivative of $\mathcal{R}_1^0$ due to motion in Σ_1), & it is expressed w.r.t. frame Σ_0 .

def ω_{12}^1 : angular velocity of frame Σ_2 that results due to change in $\mathcal{R}_2^1$, & it is expressed w.r.t. frame Σ_1 .

For the second term:

$$\begin{aligned}
 \mathcal{R}_1^0 \dot{\mathcal{R}}_2^1 &= \mathcal{R}_1^0 S(\omega_{12}^1) \mathcal{R}_2^1 \\
 &= \underbrace{\mathcal{R}_1^0 S(\omega_{12}^1) \mathcal{R}_1^{0T}}_{\substack{\text{"} \\ S(\mathcal{R}_1^0 \omega_{12}^1)}} \mathcal{R}_1^0 \mathcal{R}_2^1 \quad (\mathcal{R} S(a) \mathcal{R}^T = S(\mathcal{R}a)) \\
 &= S(\mathcal{R}_1^0 \omega_{12}^1) \mathcal{R}_1^0 \mathcal{R}_2^1 \\
 &= S(\underbrace{\mathcal{R}_1^0 \omega_{12}^1}_{\downarrow}) \mathcal{R}_2^0 \quad \dots \textcircled{4}
 \end{aligned}$$

Can be considered as the angular velocity vector ω_{12}^1 , expressed w.r.t. the coordinate frame Σ_0 .

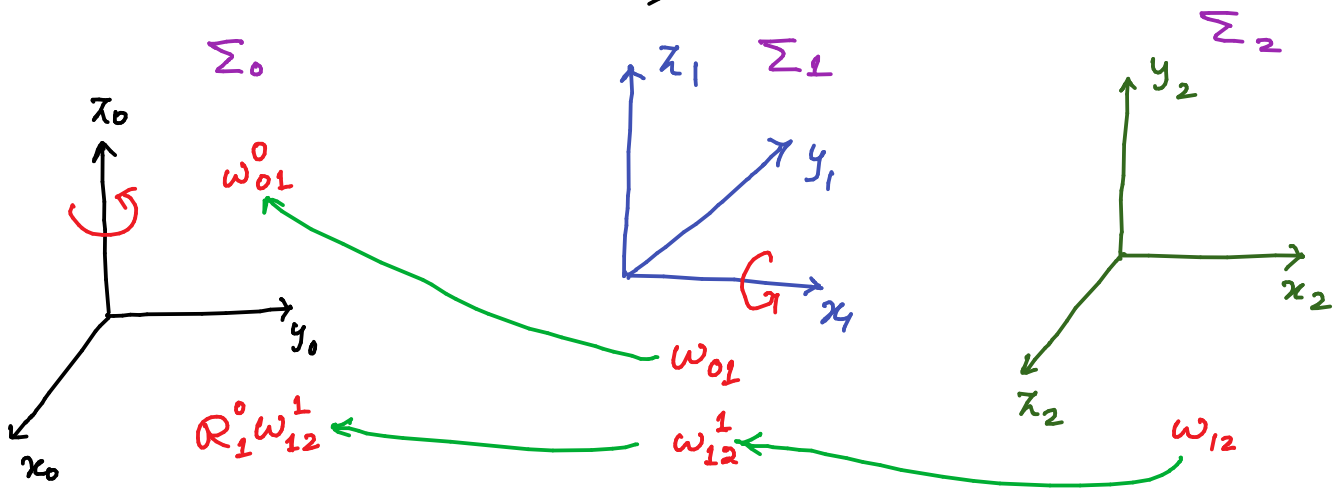
• From eqⁿ ①, ②, ③ & ④, we have:

$$\dot{\mathcal{R}}_2^0 = \dot{\mathcal{R}}_1^0 \mathcal{R}_2^1 + \mathcal{R}_1^0 \dot{\mathcal{R}}_2^1$$

$$\begin{aligned}
 \Rightarrow S(\omega_{02}^0) \mathcal{R}_2^0 &= S(\omega_{01}^0) \mathcal{R}_2^0 + S(\mathcal{R}_1^0 \omega_{12}^1) \mathcal{R}_2^0 \\
 &= \left[S(\omega_{01}^0) + S(\mathcal{R}_1^0 \omega_{12}^1) \right] \mathcal{R}_2^0 \\
 &= \left[S(\omega_{01}^0 + \mathcal{R}_1^0 \omega_{12}^1) \right] \mathcal{R}_2^0
 \end{aligned}$$

$$\Rightarrow \boxed{\omega_{02}^0 = \omega_{01}^0 + \mathcal{R}_1^0 \omega_{12}^1}$$

- The expression $\omega_{02}^0 = \omega_{01}^0 + R_1^0 \omega_{12}^1$ tells that the angular velocity vectors can be added when they are expressed relative to the same coordinate frame (in this case Σ_0).



- We can extend this idea for n -link manipulator.

$$R_n^0 = R_1^0 R_2^1 R_3^2 \dots R_n^{n-1}$$

Hence

$$\dot{R}_n^0 = S(\omega_{0n}^0) R_n^0$$

where

$$\begin{aligned} \omega_{0n}^0 &= \omega_{01}^0 + R_1^0 \omega_{12}^1 + R_2^0 \omega_{23}^2 + \dots + R_{n-1}^0 \omega_{(n-1)n}^{n-1} \\ &= \omega_{01}^0 + \omega_{12}^0 + \omega_{23}^0 + \dots + \omega_{(n-1)n}^0 \end{aligned}$$

Verify this *

• Linear Velocity of a Point Attached to a Moving Frame :

Let p be a point rigidly attached to a moving frame Σ_1 which is rotating relative to frame Σ_0 . Then

the coordinates of p w.r.t. Σ_0 is:

$$p_0 = R_1^0(t) p_1$$

Note that p_0 is time-varying since the frame Σ_1 is moving w.r.t. Σ_0 . However, p_1 is not time-varying since it is sitting on frame Σ_1 . Hence:

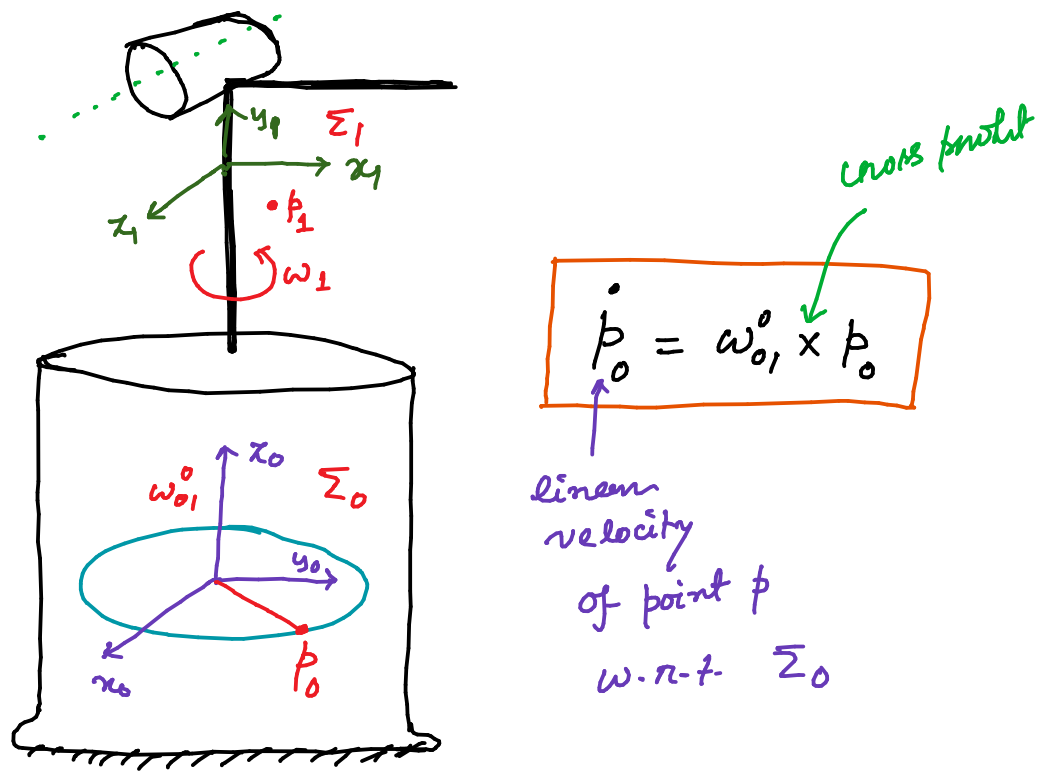
$$p_0(t) = R_1^0(t) p_1$$

$$\Rightarrow \dot{p}_0(t) = \dot{R}_1^0(t) p_1 + \underbrace{R_1^0 \dot{p}_1}_{=0}$$

$$= S(\omega_{01}^0) R_1^0(t) p_1$$

$$= S(\omega_{01}^0) p_0$$

$$= \omega_{01}^0 \times p_0$$



- Now assume the motion of Σ_1 is combination of translation & rotation w.r.t. frame Σ_0 .



Let the corresponding homogenous transformⁿ matrix be $H_1^0(t)$ be as follows, which is also time-varying :

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & d_1^0(t) \\ 0 & I \end{bmatrix}$$

Then, bringⁿ a relatⁿ between coordinates p_0 & p_1 of p throw $H_1^0(t)$ recall that we need to augment vectors p_0 & $p_1 \dots$

$$\tilde{p}_0 = \begin{bmatrix} p_0 \\ 1 \end{bmatrix} \quad \& \quad \tilde{p}_1 = \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

Then

$$\begin{aligned} \tilde{p}_0 &= H_1^0(t) \tilde{p}_1 \\ &= \begin{bmatrix} \mathcal{R}_1^0(t) & d_1^0(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \boxed{p_0 = \mathcal{R}_1^0 p_1 + d_1^0}$$

Taking derivative

$$\begin{aligned} \dot{p}_0 &= \dot{\mathcal{R}}_1^0 p_1 + \dot{d}_1^0 \\ &= S(\omega_{01}^0) \mathcal{R}_1^0 p_1 + \dot{d}_1^0 \\ &= \omega_{01}^0 \times p_0 + \dot{d}_1^0 \\ &= \omega_{01}^0 \times r + v \end{aligned}$$

where $r = p_0 = \mathcal{R}_1^0 p_1$ is the vector from origin O_1 to point p expressed in the orientation of Σ_0 , & $v = \dot{d}_1^0$ is the rate at which the origin O_1 is moving (translational motion).