

# ELL 406

## ROBOT KINEMATICS

### Part - 4

Consider a manipulator of  $n$ -links with joint variables:

$$\{q_1, q_2, \dots, q_n\}$$

Let the homogeneous transformation matrix from end-effector to the base frame is:

$$\Sigma_n \quad \Sigma_0$$

$$H_n^0(q) = \begin{bmatrix} R_n^0(q) & d_n^0(q) \\ 0 & 1 \end{bmatrix} \quad \text{where } q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

- When the robot manipulator moves, the parameters  $q_i \rightarrow q_i(t)$   
 $R(q) \rightarrow R(t)$   
 $d(q) \rightarrow d(t)$  } All will become time-varying



\* Establish a relationship between

linear/angular velocity of the end-effector to the joint velocities  $\dot{q}_i$ .

Recall that (for addition of angular velocities):

$$\dot{R}_n^0 = S(\omega_{0n}^0) R_n^0$$

$$\Rightarrow S(\omega_{0n}^0) = \dot{R}_n^0 (R_n^0)^T$$

↑ angular velocity  
of end-effector w.r.t. base  
frame

Let  $v_{0n}^0 = \dot{d}_n^0$  is the linear velocity  
of end-effector w.r.t. base  
frame.

We need expressions of the form:

$$\begin{aligned} v_{0n}^0 &= J_v \dot{q} \\ \omega_{0n}^0 &= J_\omega \dot{q} \end{aligned}$$

where  $J_v \in \mathbb{R}^{3 \times n}$  &  $J_\omega \in \mathbb{R}^{3 \times n}$  matrices

called the "manipulator Jacobian  
matrices".

$$\underbrace{\begin{bmatrix} v_{0n}^0 \\ \omega_{0n}^0 \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} J_v \\ J_\omega \end{bmatrix}}_J \dot{q}$$

$J$ : Jacobian  
matrix  
of  
the manipulator.

$$J \in \mathbb{R}^{6 \times n}$$

## → Derivation for $J_w$ :

Recall the notation:

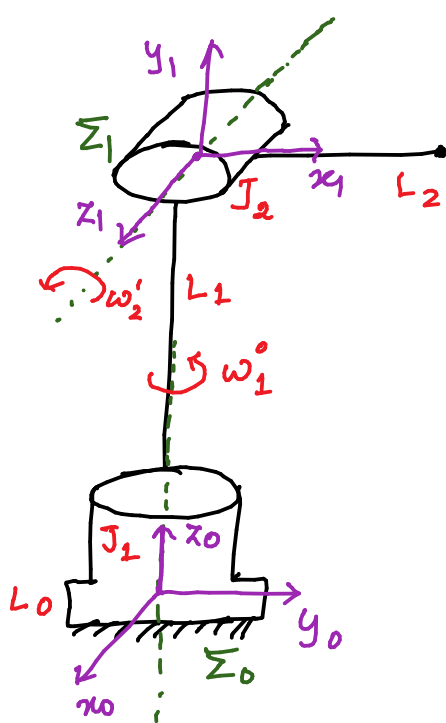
$\omega_{01}^0$ : angular velocity of frame  $\Sigma_1$ ,  
that is link-1, results due to  
change in  $R_1^0$ , & it is expressed  
w.r.t. frame  $\Sigma_0$ , i.e. link 0.

Reunite as

$\omega_1^0$ : angular velocity of link-1,  
expressed relative to (w.r.t.)  
frame  $\Sigma_0$  on link-0.

$\omega_i^{i-1}$ : angular velocity of link- $i$ ,  
produced due to rotation of joint  $i$ ,  
expressed relative to frame  $\Sigma_{i-1}$   
on link- $i-1$ .

- Recall that according to DH convention,  
we assign  $Z_i$ -axis for link- $i$  based  
of the axis of rotation/displacement  
of joint- $i$ .



Hence we can write

$$\omega_i^{i-1} = \dot{q}_i z_{i-1}$$

$$z_{i-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

unit vector in the direction of rotation.

For instance, for revolute  $J_0$ :

$$\omega_1^0 = \dot{\theta}_1 z_0$$

angular velocity of  $L_1$  obtained due to motion of joint  $J_1$ , expressed w.r.t.  $\Sigma_0$ .

\* Note that for prismatic joint the angular displacement is zero. & hence angular velocity  $\omega_i^{i-1} = 0$ .

- Recall that the angular velocity of end-effector w.r.t. base frame  $\Sigma_0$  is obtained as follows:

$$\omega_{0n}^0 = \omega_{01}^0 + R_1^0 \omega_{12}^1 + R_2^0 \omega_{23}^2 + \dots + R_{n-1}^0 \omega_{(n-1)n}^{n-1}$$

By writing  $\omega_{i-1,i}^{i-1}$  as  $\omega_i^{i-1}$ , we have

$$\omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2 + \dots + R_{n-1}^0 \omega_n^{n-1}$$

$$= p_1 \dot{q}_1 z_0 + p_2 R_1^0 \dot{q}_2 z_1 + p_3 R_2^0 \dot{q}_3 z_2 + \dots + p_n R_{n-1}^0 \dot{q}_n z_{n-1}$$

where  $p_i = \begin{cases} 1, & \text{if } J_i \text{ is revolute} \\ 0 & \text{if } J_i \text{ is prismatic} \end{cases}$

$$q_i = \theta_i \text{ for revolute } J_i$$

By denoting unit vector:  $u_{i-1} := R_{i-1}^0 z_{i-1}$

for  $i = 2, 3, \dots, n$

we have:

$$= p_1 \dot{q}_1 z_0 + p_2 \dot{q}_2 u_1 + p_3 \dot{q}_3 u_2 + \dots + p_n \dot{q}_n u_{n-1}$$

$$\Rightarrow \omega_n^0 = \underbrace{\begin{bmatrix} 1 & & & & \\ p_1 z_0 & p_2 u_1 & p_3 u_2 & \dots & p_n u_{n-1} \\ & & & & \end{bmatrix}}_{J_w} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}}_{\dot{q}}$$

$$\Rightarrow \boxed{\omega_n^0 = J_w \dot{q}}$$

→ Computation for  $J_v$ :

The linear velocity of end-effector is :  $v_n^0 = \dot{d}_n^0(q)$  i.e.

$$\dot{d}_n^0(q) = \frac{\partial d_n^0}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial d_n^0}{\partial q_2} \cdot \frac{dq_2}{dt} + \dots + \frac{\partial d_n^0}{\partial q_n} \frac{dq_n}{dt}$$

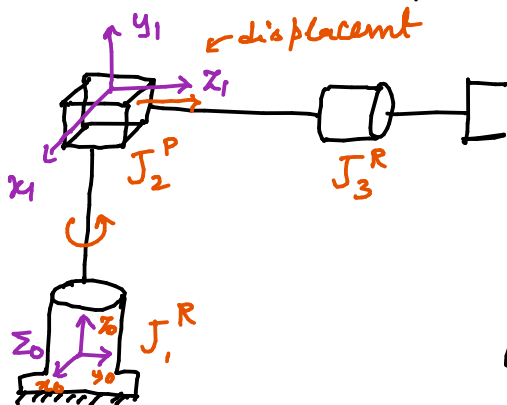
$$= \begin{bmatrix} \frac{\partial d_n^0}{\partial q_1} & \frac{\partial d_n^0}{\partial q_2} & \dots & \frac{\partial d_n^0}{\partial q_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$J_v \rightarrow$  each column is

denoted as  $J_{v_i}$

$$\Rightarrow \boxed{v_n^0 = \dot{d}_n^0 = J_v \dot{q}}$$

- Assume that all the joints of a manipulator are fixed, except  $i^{th}$  joint, which is prismatic.



$J_i^P$ : Prismatic joint

$J_i^R$ : Revolute joint

Note that here all joints are fixed &  $J_2^P$  is actuated.

Then the end-effector will be displaced along the direction of  $x_1$ -axis.

- Hence, if the  $i^{\text{th}}$  prismatic joint  $J_i^p$  is actuated, keeping others joints fixed, then  $J_i^p$  will impart a pure translation to the end-effector in the direction of  $z_{i-1}$ .



This way one can simplify <sup>2</sup>

show that  $J_{v_i} = z_{i-1}$  for prismatic joint.

Similarly  $J_{v_i} = z_{i-1} \times (O_n - O_{i-1})$ , for revolute joint.

see the details in the suggested Ref. book.

Without simplification one can just go with the relation once  $d_n^0$  is obtained.

$$J_v = \dot{d}_n^0 = \begin{bmatrix} - \frac{\partial d_n^0}{\partial q_i} \end{bmatrix} \dot{q}_i$$

J

• Hence, we have:

$$\begin{bmatrix} v_n^o \\ w_n^o \end{bmatrix} = \underbrace{\begin{bmatrix} J_v \\ J_w \end{bmatrix}}_J \dot{q}$$

also called  
"Geometric Jacobian"

$J$  is called Jacobian matrix  
of the manipulator.

→ Singular Manipulator Configuration :

↳ Singularities in the Jacobian

The Jacobian matrix  $J$  in the relat<sup>n</sup>:

$$\underbrace{\begin{bmatrix} v_n^o \\ w_n^o \end{bmatrix}}_{\xi} = \begin{bmatrix} J \end{bmatrix} \begin{bmatrix} \dot{q} \end{bmatrix} \dots (*)$$

is a mapping

$$\text{i.e. } J : \dot{q} \rightarrow \xi$$

$$\text{Jacobian } J \in \mathbb{R}^{6 \times n}$$

From the relat<sup>n</sup> (\*) it can be said that  
the all possible end-effector velocities  
are linear combinations of the columns  
of Jacobian matrix.

i.e.

$$\dot{\xi} = j_1 \dot{q}_1 + j_2 \dot{q}_2 + \dots + j_n \dot{q}_n$$

where  $J = \begin{bmatrix} | & | & & | \\ j_1 & j_2 & \dots & j_n \\ | & | & & | \end{bmatrix}$

- Since  $\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} \in \mathbb{R}^6$ , it is necessary that  $J$  must have 6 - linearly independent columns to achieve any arbitrary velocity for the end-effector.



$J$  must have rank 6

$$\boxed{\text{rank}(J) = 6}$$

- Since  $J \in \mathbb{R}^{6 \times n}$ , we have

$$\boxed{\text{rank}(J) \leq \min(6, n)}$$

For instance, for a two-link ( $n=2$ ) planar manipulator

$$\text{rank}(J) \leq 2$$

Similarly for a 3-DOF arm with a 3-DOF spherical wrist manipulator:

$$\text{rank}(J) \leq 6$$

- Hence, rank of  $J$  depends on the configuration variable or joint variable  $q$ .
- The configuration for which  $\text{rank}(J)$  is less than its maximum value are called "singular configuration" of a manipulator.
- For example, 6-DOF manipulator with spherical wrist:  $J \in \mathbb{R}^{6 \times 6}$ . Hence the singular configurations are those  $q$  where  $\boxed{\det J(q) = 0}$ .

## → Inverse Velocity Kinematics :

The relationship :

$$\textcircled{A} \quad \left\{ \begin{array}{l} \underbrace{\begin{bmatrix} v_n^o \\ \omega_n^o \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} J_v \\ J_w \end{bmatrix}}_J \begin{bmatrix} \dot{q} \end{bmatrix} \end{array} \right. \quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

↙ Needs to determine

provides the end-effector's velocity  
that results due to joint velocities  $\dot{q}_i$ .

- For the inverse problem :

" Finding of joint velocities  $\dot{q}_i$   
that produces desired end-effector  
velocity  $\xi$  ".

- From the relation  $\textcircled{A}$ , it is easy  
to see that when Jacobian matrix  $J$   
is square & non-singular, the  
joint velocities computation is very  
easy i.e.

$$\dot{q} = J^{-1} \xi$$

→ The manipulators having joints less than 6, the Jacobian can not be inverted. In that case, the sol<sup>n</sup> to

$$\dot{\mathbf{x}} = \mathbf{J} \dot{\mathbf{q}}$$

exists if & only if  $\dot{\mathbf{x}} \in \text{Im}(\mathbf{J})$

Image space of  $\mathbf{J}$ .

• A vector  $\dot{\mathbf{x}} \in \text{Im}(\mathbf{J})$  if and only if

$$\text{rank}(\mathbf{J}) = \text{rank}([\mathbf{J} : \dot{\mathbf{x}}])$$

$\mathbf{J} \in \mathbb{R}^{6 \times n}$   
↑  
No of joints

↑  
augmented matrix

• For the case when  $n > 6$ , we can solve for  $\dot{\mathbf{q}}$  using right pseudo-inverse of  $\mathbf{J}$ .

• For a full row-rank matrix  $\mathbf{J} \in \mathbb{R}^{6 \times n}$ ,

the matrix :  $\mathbf{J}\mathbf{J}^T$  is invertible

↓

$(\mathbf{J}\mathbf{J}^T)^{-1}$  exists.

By uniting:  $I = (JJ^T)(JJ^T)^{-1}$

$$= J \left[ \underbrace{J^T (JJ^T)^{-1}}_{J^+} \right]$$

$$= JJ^+ \quad \begin{matrix} \uparrow \\ \text{Right} \end{matrix}$$

pseudo-inverse of  $J$ .

Since  $JJ^+ = I$

However,  $J^+J \neq I$ .

def

$$\dot{q} = J^+ \xi + (I - J^+J) b \quad \dots \quad (1)$$

where  $b \in \mathbb{R}^n$  is any vector.

Then,

$$J \dot{q} = J [J^+ \xi + b - J^+J b]$$

$$= \underbrace{JJ^+}_{I} \xi + Jb - \underbrace{JJ^+}_{I} Jb$$

$$= \xi + Jb - Jb$$

$$= \xi$$

$\Downarrow$

The  $\dot{q}$  in (1) is a solution to

$$\xi = J \dot{q} \quad \text{relation.}$$

- Computation of  $J^+$  :

Let  $J$  has singular value decomposition:

$$J = U \Sigma V^T$$

where  $U \in \mathbb{R}^{6 \times 6}$   $V \in \mathbb{R}^{n \times n}$  are orthogonal matrices &

$J$  is  
full row  
rank.

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_6 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$\sigma_i$  :  $i$ th singular value of  
Jacobian matrix  $J$ .

• Then

$$J^+ = V \Sigma^+ U^T$$

where

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & & & \\ & 1/\sigma_2 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_6 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}^T$$

→ Intuition behind the choice of  $\dot{q}$  as in (1)

We explain using solving of set of linear equations:

$$Ax = b \quad \text{for } A \in \mathbb{R}^{m \times n} \text{ is full row rank.}$$

$J\dot{q} = \frac{b}{2}$

Let us write the sol<sup>n</sup>  $x$  of  $Ax$  as:

$$x = x_n + x_n^\perp$$

where  $x_n \in \mathcal{R}(A)$  (Row space of  $A$ )

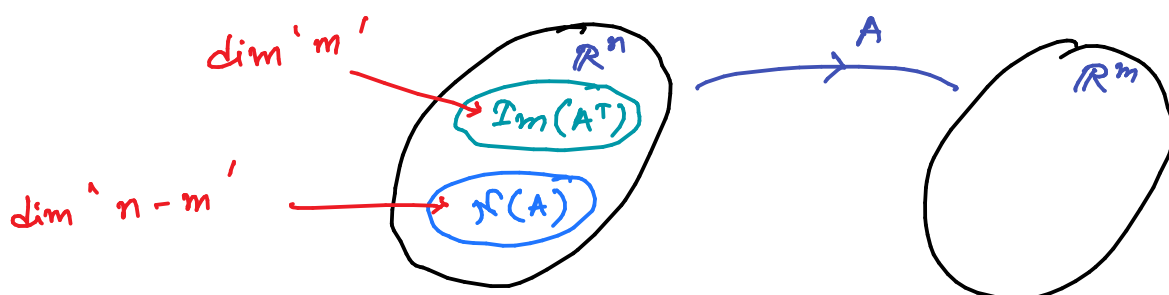
$x_n^\perp \in [\mathcal{R}(A)]^\perp$  orthogonal complement of  $\mathcal{R}(A)$

From lin. algebra results:

$$\begin{aligned} \mathcal{R}(A) &= \text{Im}(A^T) \\ \mathcal{N}(A) &= [\mathcal{R}(A)]^\perp \end{aligned}$$

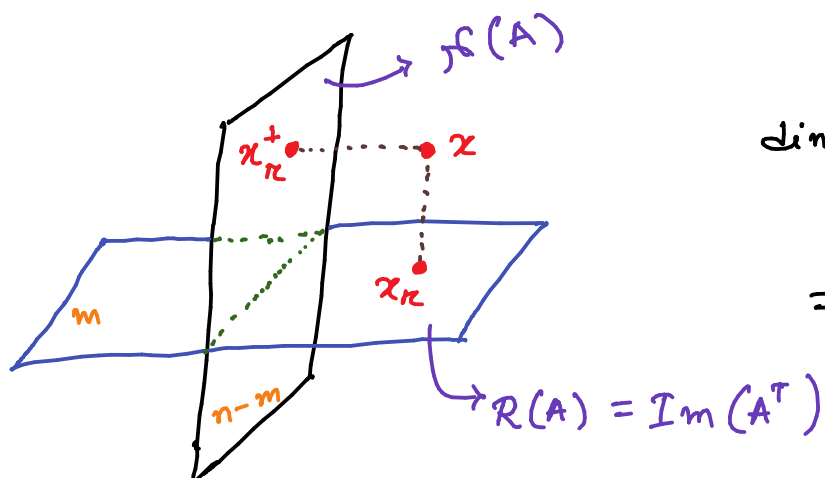
$\text{Im}(\cdot)$ : Image space of  $A$

$\mathcal{N}(\cdot)$ : Null space of  $A$



Since  $x_n \in \mathcal{R}(A) = \text{Im}(A^T)$

$$\text{and } x_n^\perp \in \mathcal{N}(A)$$



$$\dim[\text{Im}(A^T)] + \dim[\mathcal{N}(A)]$$

$$= m + n - m = n$$

$$x \in \mathbb{R}^n$$

$$x = x_n + x_n^\perp$$

Since  $Ax = b$

we have  $A(x_n + x_n^\perp) = b$

$$\Rightarrow Ax_n + \underbrace{Ax_n^\perp}_{=0} = b$$

$$\textcircled{2} \quad \dots \Rightarrow \boxed{Ax_n = b} \quad x_n \in \mathbb{R}^n$$

Further, since  $x_n \in \text{Im}(A^T) = \mathcal{R}(A)$ ,

$\Downarrow$

$\exists$  a vector  $\eta \in \mathbb{R}^m$

$$A^T \eta = x_n$$

$$A^T \eta = x_n$$

$$\Rightarrow A A^T \eta = A x_n$$

$$\Rightarrow (A A^T) \eta = b \quad (\text{from } \textcircled{2})$$

$$\Rightarrow \eta = (A A^T)^{-1} b \quad (\text{since } A \text{ is full row rank})$$

$$\Rightarrow A^T \eta = A^T (A A^T)^{-1} b$$

$$\Rightarrow x_n = \underbrace{A^T (A A^T)^{-1}}_{A^+} b$$

$$\Rightarrow \boxed{x_n = A^+ b} \leftarrow \begin{array}{l} \text{Characterization} \\ \dots \textcircled{3} \text{ of } x_n \text{ in terms} \\ \text{of } A \text{ \& } b. \end{array}$$

• Now since  $x_n^+ \in \mathcal{N}(A)$

$\Downarrow$

$$A x_n^+ = 0$$

Note that for any  $y \in \mathbb{R}^n$ ,

$$\text{we have } A y - A y = 0$$

which can be written as

$$A y - (A A^T)(A A^T)^{-1} A y = 0$$

Then rearranging

$$A \left[ y - \underbrace{A^T (A A^T)^{-1} A y}_{A^+} \right] = 0$$

Hence  $A [y - A^+ A y] = 0$

$$\Downarrow$$

$$(y - A^+ A y) \in \mathcal{N}(A)$$

$$\Downarrow$$

We can set

$$\begin{aligned} x_n^\perp &= y - A^+ A y \\ &= (I - A^+ A) y \end{aligned}$$

Hence the sol<sup>n</sup> of  $Ax = b$  is

$$\begin{aligned} x &= x_n + x_n^\perp \\ &= A^+ b + (I - A^+ A) y \end{aligned}$$

• Now comparing with our velocity kinematics eq<sup>n</sup> :  $J \dot{q} = \xi$

The sol<sup>n</sup> set is

$$\begin{aligned} \dot{q} &= \dot{q}_n + \dot{q}_n^\perp \\ &= J^+ \xi + (I - J^+ J) b \end{aligned}$$

for any vector  $b \in \mathbb{R}^n$ .

- Now note that

$$\dot{q}_n^\perp = (I - J^+ J) b \in \mathcal{N}(J)$$



Null space of  
the Jacobian matrix



If  $\dot{q}'$  is a joint velocity s.t.

$$\boxed{\dot{q}' = (I - J^+ J) b}, \text{ then when}$$

the joints move with velocity  $\dot{q}'$ ,  
the end-effector will remain Fixed.

Since  $\dot{q}' \in \mathcal{N}(J)$ .



The end-effector will remain  
fixed even if there is joint  
velocities.

- Further, observe that for the following minimum norm optimization problem:

$$\begin{cases} \min \|x\| \\ \text{s.t. } Ax = b \end{cases}$$

where  $A \in \mathbb{R}^{m \times n}$  is full row rank matrix

Solution to this problem:

All sol<sup>n</sup> of  $Ax = b$  are:

$$\begin{aligned} x &= x_n + x_n^\perp \\ &= A^+ b + (I - A^+ A) y \end{aligned}$$

for all  $y \in \mathbb{R}^n$ .

Hence min. norm sol<sup>n</sup> is obtained by choosing  $y = 0$ .

Then, the solution to the optimization is:

$$x_{\min} = A^+ b = A^T (A A^T)^{-1} b$$

- Hence, the min. norm joint velocity is:

$$\dot{q}_{\min} = J^+ \dot{\xi}$$

