

- The objective of path planning (PP) & trajectory planning (TP) is to generate the reference inputs to the motion control system for the manipulator so that the manipulator traversed through the planned path & trajectories.

→ Path :

A path is a locus of points in the robot work-space, which the manipulator has to follow in the execution of the assigned motion.



It is purely a geometric description of motion. It does not specify what should be the joint v & accelerations while traversing the path.

- Trajectory :

A trajectory is a path on which a timing law is specified, for instance, in terms of velocities and/or accelerations at each point in the path.

→ Path Planning :

- Configuration Space :

The vector of joint variables q_i

$q = [q_1, q_2, \dots, q_n]^T$ provides a representation of a configuration of a manipulator.

↓

- The set of all $q \in \mathbb{R}^n$ is referred to as "configuration space: Q ".

↓

There are several other ways to represent a configuration space.

Some notations:

- W : Manipulator workspace
- $\bar{O}_i$: An obstacle in the workspace
- $A(q)$: Robot, which is a subset of workspace occupied by the robot at configuration q .

→ To plan a collision-free path, we need to ensure that the robot never reaches to a configuration q that can make contact with an obstacle.

- The set of configurations for which the robot collides with an obstacle is referred to as the "Configuration space obstacle".


$$Q_0 = \{ q \in Q \mid A(q) \cap \bar{O} \neq \emptyset \}$$

where $\bar{O} = \bigcup_i \bar{O}_i$

- The space of collision-free configuration space $Q_f = Q \setminus Q_0$

→ Path Planning Using Potential Fields:

- Explicit representation or characterization of Q_0 & Q_f are not feasible.



We use a search algorithm that incrementally explores Q_f in the path planning.



Artificial Potential field method searching for Q_f .



$$U(q) = U_a(q) + U_r(q)$$



Attractive potential field that attracts robots from its starting configuration q_s to the final configuration q_f .



Repulsive potential field that repels the robot from the boundary of Q_0 .

→ Recall that for an n -link manipulator (excluding base link L_0), we can assign n -coordinate frames: $\Sigma_1, \Sigma_2 \dots \Sigma_n$ according to DH-convention.



There are ' n ' origins $O_1, O_2 \dots O_n$ corresponding to each Σ_i .



We define potential field for each of the origins of the n -coordinate frames.

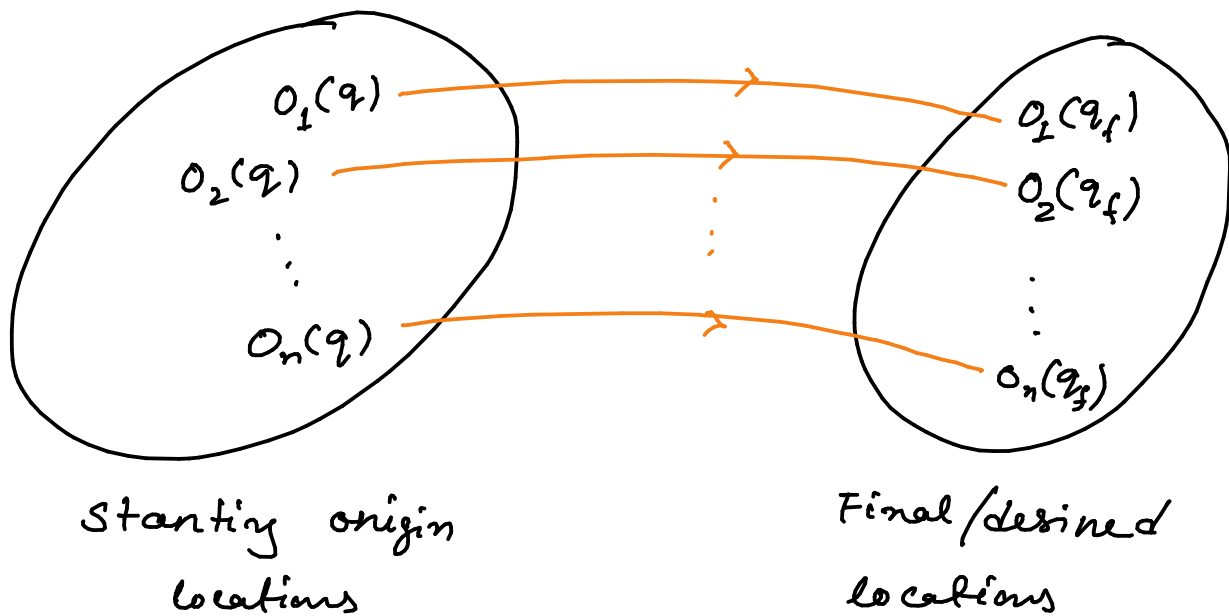


- These workspace potential fields will attract the n -origins to their final locations while repelling them from obstacles.

→ Attractive Potential Field:

We define attractive potential fields for each n -origin O_i such that they will reach to the final/desired origins.

Once the origins are reached to final positions the robot will have reached to its final configuration.



- Each origin of the n -frames need to transfer to one coordinate frame, preferably to the base-frame Σ_0 . Hence each origin O_i is a function of joint parameter $q \rightarrow$ represented as $O_i(q)$.
- Attractive potential field:

$$U_{a_i}(q) := \frac{1}{2} \gamma_i \|O_i(q) - O_i(q_f)\|_2^2$$

where γ_i is a scaling factor.



Sometimes referred to as "Parabolic well Potential"

We can now frame an optimization problem to minimize $U_{a_i}(q)$ for the path planning problem.

- One method to solve the optimization is using "Gradient Descent" method.

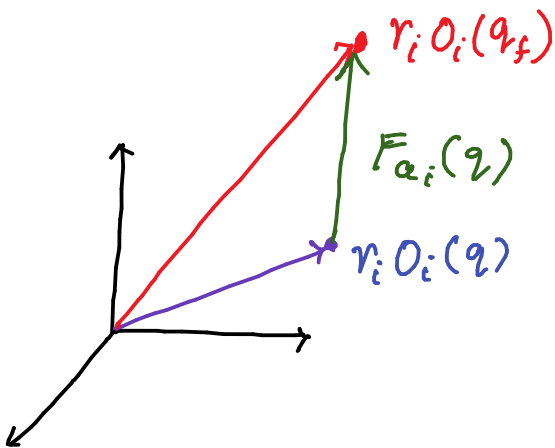


The negative gradient of U can be considered as a force acting on the robot in the configuration space.

- Hence the attractive force :

$$F_{a_i}(q) = -\nabla U_{a_i}(q) = -r_i \left[O_i(q) - O_i(q_f) \right]$$

↑ Gradient of U_{a_i}
↑ verify

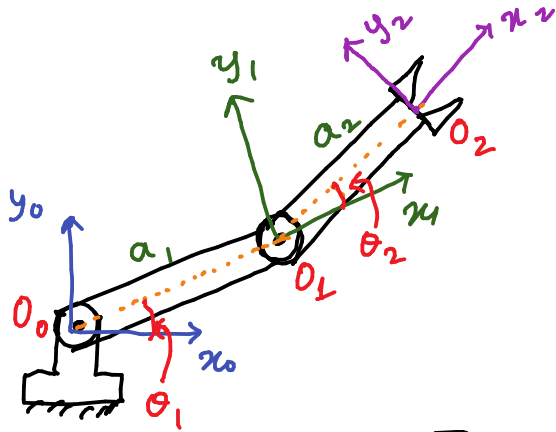


$F_{a_i}(q)$ is directed towards $O_i(q_f)$.

- If q is far away from q_f the above attractive force $F_{a_i}(q)$ will be very high.

Example

Consider a 2-link manipulator:



DH - Table

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

The homogeneous transformation matrices are:

$$H_1 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} R_2^0 & d_2^0 \\ 0 & 1 \end{bmatrix}$$

where $H_i = H_{x_i \theta_i}^R H_{x_i d_i}^T H_{x_i \alpha_i}^T H_{x_i a_i}^R$

Then $H_1^0 = H_1 \quad \text{and} \quad H_2^0 = H_1 H_2$

Recall from previous computation [Example - 3.1 in

Spong & Vidyasagar's
"Robot Modeling and Control" book]

$$H_1^0 = H_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} s_1 &= \sin \theta_1 \\ c_1 &= \cos \theta_1 \end{aligned}$$

$$H_2^0 = H_1 H_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_{12} + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_{12} + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_{12} = \cos(\theta_1 + \theta_2) \quad s_{12} = \sin(\theta_1 + \theta_2)$$

We know:

when a point p_1 w.r.t. Σ_1 needs to be represented w.r.t. Σ_0 as p_0 then following transformⁿ needs to be performed:

$$\begin{bmatrix} p_0 \\ 1 \end{bmatrix} = H_1^0 \begin{bmatrix} p_1 \\ 1 \end{bmatrix} \dots \textcircled{1} \quad H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

Similarly for p_i w.r.t. Σ_i represented as p_0 w.r.t. Σ_0 through:

$$\begin{bmatrix} p_0 \\ 1 \end{bmatrix} = H_i^0 \begin{bmatrix} p_i \\ 1 \end{bmatrix}$$

$$\text{Hence} \quad \begin{bmatrix} p_0 \\ 1 \end{bmatrix} = H_2^0 \begin{bmatrix} p_2 \\ 1 \end{bmatrix} \dots \textcircled{2} \quad H_2^0 = \begin{bmatrix} R_2^0 & d_2^0 \\ 0 & 1 \end{bmatrix}$$

When $p_i = o_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\uparrow$ origin of i th coordinate frame.

Then according to ①, we have

$$\begin{bmatrix} p_0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} o_1 \\ 1 \end{bmatrix}$$

H_1^0

$$= \begin{bmatrix} d_1^0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{p_0 = d_1^0} \dots$$

$\Downarrow$

When origin O_1 referred to Σ_0 it becomes d_1^0 .

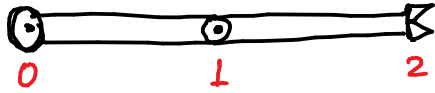
Similarly when O_2 referred to Σ_0 it becomes d_2^0 .

• In the above 2-link example

$$O_1 = d_1^0 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \end{bmatrix}$$

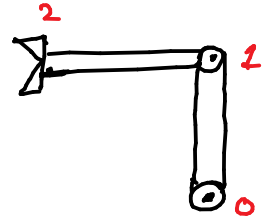
$$O_2 = d_2^0 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \end{bmatrix}$$

Starting manipulator configuration



$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Desired/final configuration



$$q_f = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 90^\circ \\ 90^\circ \end{bmatrix}$$

For $a_1 = a_2 = 1$

$$O_1(q) = \begin{bmatrix} a_1 \cos 0^\circ \\ a_1 \sin 0^\circ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$O_1(q_f) = \begin{bmatrix} a_1 \cos 90^\circ \\ a_1 \sin 90^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$O_2(q) = \begin{bmatrix} a_1 \cos 0^\circ + a_2 \cos 0^\circ \\ a_1 \sin 0^\circ + a_2 \sin 0^\circ \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$O_2(q_f) = \begin{bmatrix} a_1 \cos 90^\circ + a_2 \cos 180^\circ \\ a_1 \sin 90^\circ + a_2 \sin 180^\circ \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Hence } F_{a_1}(q) &= -r_1 [O_1(q) - O_1(q_f)] \\ &= r_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$F_{a_2}(q) = -r_2 [O_2(q) - O_2(q_f)]$$

$$= r_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

r_1 & r_2 are free scaling parameters.

→ Repulsive Field :

↳ prevent collisions between robot & obstacles.

↓

Nature of the potential field

- Its value approaches to ∞ as the configuration approaches to an obstacle boundary
- Its value decreases to zero at a specified distance from the obstacle boundary

Let

ρ_0 : An obstacle will not repel origin O_i if the distance from O_i to obstacle greater than ρ_0 .

- One such repulsive potential field:

$$U_{\pi i}(q) := \begin{cases} \frac{1}{2} \eta_i \left(\frac{1}{\varphi(O_i(q))} - \frac{1}{\varphi_0} \right)^2, & \text{for } \varphi(O_i(q)) \leq \varphi_0 \\ 0 & \text{for } \varphi(O_i(q)) > \varphi_0 \end{cases}$$

where $\varphi(O_i(q))$: shortest distance between O_i & any obstacle in the workspace.

η_i : scaling factor

- Then the workspace repulsive force is equal to the negative gradient of $U_{\pi i}(q)$

i.e.

$$F_{\pi i}(q) = -\nabla U_{\pi i}(q)$$

$$F_{\pi i}(q) = \eta_i \left[\frac{1}{\varphi(O_i(q))} - \frac{1}{\varphi_0} \right] \cdot \frac{1}{\varphi^2(O_i(q))} \cdot \nabla \varphi(O_i(q))$$

where $\nabla \varphi(O_i(q))$ is the gradient of function $\varphi(x)$ evaluated at $x = O_i$

If the obstacle region is convex set

& b is the point on the obstacle boundary that is closest to O_i , then

$$\varphi(O_i(q)) = \|O_i(q) - b\|$$

& then its gradient

$$\nabla \varphi(O_i(q)) = \frac{O_i(q) - b}{\|O_i(q) - b\|}$$



The unit vector
directed from b towards
 O_i .

→ Recall that the potential field for the robot path planning is

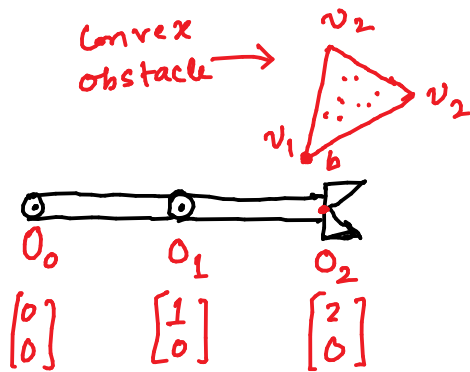
$$U(q) = U_a(q) + U_r(q)$$

As discussed above, the force on the robot in the configuration space is the negative of gradient of $U(q)$ i.e.

$$F(q) = -\nabla U(q) = -\nabla U_a(q) - \nabla U_r(q)$$

Example

Consider the 2-link manipulator, considered before.



The nearest obstacle point to the manipulator is vertex v_1 of the triangular convex obstacle.

$$\text{Let } v_1 = b = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

$$\text{Recall } O_1(q) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad O_2(q) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The nearest origin to b is $O_2(q)$. Hence the shortest distance between $O_2(q)$ & b is

$$\|O_2(q) - v_1\| = \left\| \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \right\| = 0.5$$

$$\text{Hence: } \rho(O_2(q)) = 0.5.$$

$$\& \quad \nabla \rho(O_2(q)) = \frac{O_2(q) - b}{\|O_2(q) - b\|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence, the repulsive force at the initial/starting configuration for O_2 is:

$$F_{r_2}(q) = \eta_2 \left[\frac{1}{0.5} - \frac{1}{1} \right] \frac{1}{0.25} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \eta_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

Since the nearest origin to the obstacle point b is O_2 , & the distance betⁿ O_2 & b is 0.5, we can set $\rho_0 = 1$ (only tw^o strictly greater than 0.5).

• The shortest distance between O_1 & b is

$$\begin{aligned}\varphi(O_1(q)) &= \|O_1(q) - b\| = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} \right\| = \sqrt{1^2 + 0.5^2} \\ &= 1.11\end{aligned}$$

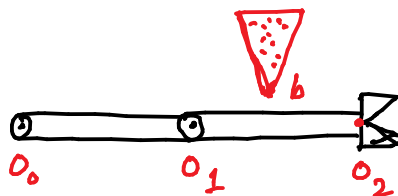
Since $\rho_0 = 1$, the origin O_1 will not see any repulsive force.

↑

The repulsive potential field is zero.

→ Note that defining repulsive fields only for the origins of the DH-frame for the manipulator may not guarantee that robot will not collide with an obstacle.

↓



In such situation we use a set of
"floating repulsive control points",
(O_{fei}) one for each link.
↑

These points are chosen on the
boundary of each link that are
closest to any workspace obstacle.

(For the previous 2-link manipulator case
 O_{fe2} can be chosen in the middle
of link - 2 : L_2 .)

The repulsive force acting on O_{fei} is
defined similar to $F_{ri}(q)$ for
origin O_i .

→ Mapping Workspace Forces to Joint Torques :

Let τ denotes the vector of joint
torques induced by the workspace force
 F exerted on the end-effector, then

$$\boxed{J_v^T F = \tau} \quad \dots \textcircled{A}$$

↑
Linear velocity Jacobian matrix.

The relation ① will be used to map the workspace force to joint torques.

Example

Consider again the same 2-link manipulator.

Recall the forward velocity kinematics relatⁿ:

$$v_n^o = J_v \dot{q}$$

linear velocity of end effector expressed w.r.t. base-frame.

$$\dot{O}_n^o = \underbrace{\begin{bmatrix} \frac{\partial O_n^o}{\partial q_1} & \frac{\partial O_n^o}{\partial q_2} & \dots & \frac{\partial O_n^o}{\partial q_n} \end{bmatrix}}_{J_v} \dot{q}$$

let's write it as

$$J_{O_n}(q)$$



This relation can be written for any O_i^o .

i.e.

$$\dot{O}_i^o = \underbrace{\begin{bmatrix} \frac{\partial O_i^o}{\partial q_1} & \frac{\partial O_i^o}{\partial q_2} & \dots & \frac{\partial O_i^o}{\partial q_n} \end{bmatrix}}_{J_{O_i}(q)} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Verify that for $a_1 = a_2 = 1$:

$$J_{0_2}(q) = \begin{bmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{bmatrix}$$

$$s_{12} = \sin(\theta_1 + \theta_2)$$

$$c_{12} = \cos(\theta_1 + \theta_2)$$

$$s_1 = \sin \theta_1$$

$$c_1 = \cos \theta_1$$

$$\text{or } J_{0_1}(q) = \begin{bmatrix} -s_1 & 0 \\ c_1 & 0 \end{bmatrix}$$

Then for $q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$J_{0_1}^T(q) = \begin{bmatrix} -s_1 & c_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J_{0_2}^T(q) = \begin{bmatrix} -s_1 - s_{12} & c_1 + c_{12} \\ -s_{12} & c_{12} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

By letting $\gamma_i = \eta_i = 1$, we have:

$$\tau_{a1}(q) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \leftarrow \text{attractive torque for joint 1.}$$

$\uparrow F_{a1}(q)$

$$\tau_{a2}(q) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\tau_{r2}(q) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} \quad \leftarrow \text{repulsive torque for Joint - 2.}$$

$\uparrow F_{r2}(q)$

→ Gradient Descent Algorithm for Path Planning :

↑ Starting at the initial configuration q_i , take a small step in the direction of negative gradient. This will produce new configuration. If this is not the final then continue till the final configuration is reached.

① $q^0 \leftarrow q_i, i=0$

② if

$$\|q^i - q_f\| > \epsilon$$

$$q^{i+1} \leftarrow q^i + \alpha^i \frac{\gamma(q^i)}{\|\gamma(q_i)\|}$$

$$i \leftarrow i+1$$

else return

$$[q^0, q^1 \dots q^i]$$

③ Go to Step-2

ϵ is a very small number

α_i : step size at each iteration

$\gamma(q_i)$: torque

choice of α_i is critical.