

PATH & TRAJECTORY PLANNINGPart - 2

- Path : γ is a continuous map

$$\gamma : [0, 1] \rightarrow \mathcal{Q}$$

↑ configuration space

s.t.

$$\begin{aligned} \gamma(0) &= q_s \leftarrow \text{starting configuration} \\ \gamma(1) &= q_f \leftarrow \text{Final configuration.} \end{aligned}$$

- Trajectory : γ is a time-varying function

$q(t)$ such that :



$$q(t_0) = q_s \quad t_0 : \text{initial time}$$

$$q(t_f) = q_f \quad t_f : \text{final time}$$

Since trajectory $q(t)$ is a function of time, we can compute velocities & accelerations of the manipulator (joints) by differentiation.

- Path planning will give us a sequence of points along the path i.e. the sequence of joint-parameters q .

- For a trajectory, we assign a time-law to the path i.e. $q(t)$ with $q(t_0) = q_s$ & $q(t_f) = q_f$, so that we can specify / get information of joint velocities & accelerations over the entire time $t \in [t_0, t_f]$.

- Often the paths are specified by a sequence of end-effector's configurations. In that case, one needs to solve the inverse-kinematics problem to get the sequence of joint variables q , corresponding to the specified end-effector's configurations.

→ Trajectory Planning:

↑ The problem is to find a trajectory that connects the initial & final configurations while satisfying constraints such as velocity and/or acceleration.

- We discuss the planning for a single joint. The same process can be followed for other joints as well.
- Let the initial time t_0 & final time t_f a joint variable have following values:

$$q(t_0) = q_0$$

$$q(t_f) = q_f$$

$$\dot{q}(t_0) = v_0$$

initial

$$\dot{q}(t_f) = v_f \leftarrow \text{velocity constraint}$$

final

- In addition to velocity constraint, we may also have acceleration constraint:

$$\ddot{q}(t_0) = a_0$$

$$\ddot{q}(t_f) = a_f$$

→ Cubic Polynomial Trajectories:

Let there are following 4 constraints:

The trajectory must satisfy all these 4 constraints

$$\left\{ \begin{array}{ll} q(t_0) = q_0 & q(t_f) = q_f \\ \dot{q}(t_0) = v_0 & \dot{q}(t_f) = v_f \end{array} \right.$$

- Since there are 4 constraints, we need a polynomial which has 4-independent coefficients, so that they can independently be chosen to satisfy these constraints.

Hence, we consider a cubic trajectory of the form:

$$\begin{aligned} q(t) = x(t) &= \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0 \\ \Rightarrow \dot{q}(t) &= 3\alpha_3 t^2 + 2\alpha_2 t + \alpha_1 \end{aligned}$$

Using them

$$\begin{cases} q_0 = \alpha_3 t_0^3 + \alpha_2 t_0^2 + \alpha_1 t_0 + \alpha_0 \\ v_0 = 3\alpha_3 t_0^2 + 2\alpha_2 t_0 + \alpha_1 \\ q_f = \alpha_3 t_f^3 + \alpha_2 t_f^2 + \alpha_1 t_f + \alpha_0 \\ v_f = 3\alpha_3 t_f^2 + 2\alpha_2 t_f + \alpha_1 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}}_{\text{given}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\text{solve}} = \underbrace{\begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}}_{\text{given}}$$

t_0 & t_f are given.

- A cubic trajectory gives continuous position & velocities at initial & final points, however, discontinuities for acceleration. The derivative of acceleration is called a jerk.

Discontinuities in the acceleration may lead to impulsive jerk, which may excite vibrational modes of the manipulator.

For this, we often need to impose extra constraints on accelerations, in addition to velocities & positions i.e.

$$\ddot{q}(t_0) = a_0 \quad \ddot{q}(t_f) = a_f$$

⇓

We have total 6 constraints

⇓

We need a time-polynomial trajectory which has 6 - coefficients:

$$q(t) = x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \alpha_5 t^5$$

↑ Quintic
Polynomial

↓

We then follow the similar process as before to compute all these 6-coefficients.

- Trapezoidal Velocity Profile Trajectory :

(Linear segments with Parabolic Blends)

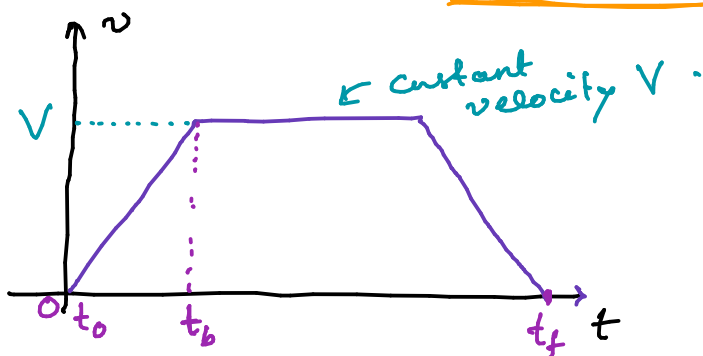
Divide the velocity profile into following three parts:

(i) Time $t_0 \rightarrow t_b$: Linear Ramp velocity
 $\uparrow$ blend time $\quad \quad \quad \uparrow$ $\alpha(t)$ is a quadratic polynomial

(ii) Time $t_b \rightarrow t_f - t_b$: Constant velocity
 is a linear function in time

(iii) Time $t_f - t_b \rightarrow t_f$: Linear velocity
 quadratic polynomial in time.

Ensure : $\dot{q}(t_0) = \dot{q}(t_f) = 0$



Desired velocity profile

Let :

$$\begin{cases} q(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 \\ \dot{q}(t) = \alpha_1 + 2\alpha_2 t \\ \ddot{q}(t) = 2\alpha_2 \end{cases}$$

We have the constraints

$$q(t_0) = q_0 \quad \dot{q}(t_0) = v_0 = 0$$

Let $t_0 = 0$

The $\underbrace{q(0)}_{q(t_0)=q_0 \text{ given}} = \alpha_0 \Rightarrow \boxed{\alpha_0 = q_0}$

$\underbrace{\dot{q}(t_0)}_{\substack{|| \\ v_0=0 \text{ given}}} = \alpha_1 \Rightarrow \boxed{\alpha_1 = 0}$

At time t_b , we need constant velocity V i.e.

$$\dot{q}(t_b) = \alpha_1 + 2\alpha_2 t_b$$

$$\Rightarrow V = 0 + 2\alpha_2 t_b$$

$$\Rightarrow \boxed{\alpha_2 = \frac{V}{2t_b}}$$

- From the above, we have the following trajectory for $t_0 \rightarrow t_b$:

for $t \in [t_0, t_b]$ $\left\{ \begin{array}{l} q(t) = q_0 + \frac{V}{2t_b} t^2 \dots \textcircled{\Delta} \\ \dot{q}(t) = \frac{V}{t_b} t \\ \ddot{q}(t) = \frac{V}{t_b} \end{array} \right.$

- For time $t_b \rightarrow t_f - t_b$:

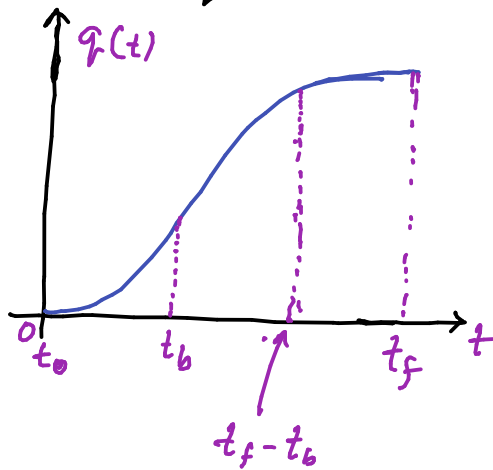
The trajectory within this time period is having a constant velocity profile V .

Hence

$$\begin{aligned} q(t) &= \alpha_0 + \alpha_1 t \\ &= q(t_b) + V(t - t_b) \quad \dots (*) \end{aligned}$$

Propose this

We have chosen t_b such that the position curve $q(t)$ is symmetric as in below fig.



Hence, by symmetry

$$q\left(\frac{t_f}{2}\right) = \frac{q_0 + q_f}{2}$$

Then the relat' (*)

becomes:

$$q\left(\frac{t_f}{2}\right) = q(t_b) + V\left(\frac{t_f}{2} - t_b\right)$$

$$\Rightarrow \frac{q_0 + q_f}{2} = q(t_b) + V\left(\frac{t_f}{2} - t_b\right)$$

$$\Rightarrow q(t_b) = \frac{q_0 + q_f}{2} - \frac{V t_f}{2} + V t_b$$

V is a scalar
of const. velocity

$$\Rightarrow q(t_b) = \frac{q_0 + q_f - V t_f}{2} + V t_b \dots \textcircled{**}$$

At time t_b the joint parameter on the position $q(t)$ must be same for

$[0, t_b]$ & $[t_b, t_f - t_b]$. Hence it follows from

① & $\textcircled{**}$ that

$$q_0 + \frac{V}{2} t_b^2 = \frac{q_0 + q_f - V t_f}{2} + V t_b$$

$$\Rightarrow q_0 + \frac{V}{2} t_b = \frac{q_0 + q_f - V t_f}{2} + V t_b$$

$$\Rightarrow \boxed{t_b = \frac{q_0 - q_f + V t_f}{V}} \dots \textcircled{1}$$

We also have the constraint

$$0 < t_b \leq \frac{t_f}{2}$$

$$\Rightarrow 0 < \frac{q_0 - q_f + V t_f}{2} \leq \frac{t_f}{2}$$

$$\Rightarrow 0 < \frac{q_0 - q_f}{2} + \frac{V t_f}{2} \leq \frac{t_f}{2}$$

$$\Rightarrow 0 < \frac{q_0 - q_f}{2V} + \frac{t_f}{2} \leq \frac{t_f}{2V}$$

by multiplying $\frac{1}{V}$

By rearranging we have following two relations:

$$\frac{q_f - q_0}{V} < t_f \leq \frac{2(q_f - q_0)}{V}$$

Decide V first, & then choose a t_f using this relⁿ.

First decide t_f , & then choose a V using this relⁿ

$$\frac{q_f - q_0}{t_f} < V \leq \frac{2(q_f - q_0)}{t_f}$$

The constant velocity must be within this limit.

Hence, for time $t_b \rightarrow t_f - t_b$ we have:

$$\left. \begin{aligned} q(t) &= q(t_b) + V(t - t_b) \\ &= \frac{q_0 + q_f - V t_f}{2} + V t_b + V t - V t_b \\ &= \frac{q_0 + q_f - V t_f}{2} + V t \\ \dot{q}(t) &= V \\ \ddot{q}(t) &= 0 \end{aligned} \right\} t \in [t_b, t_f - t_b]$$

Exercise : Using symmetric properties, derive $q(t)$, $\dot{q}(t)$ & $\ddot{q}(t)$ for $[t_f - t_b, t_f]$.

$$q(t) = \begin{cases} q_0 + \frac{V}{2t_b} t^2 & t \in [0, t_b] \\ \frac{q_f + q_0 - Vt_f}{2} + Vt & t \in [t_b, t_f - t_b] \\ q_f - \frac{Vt_f^2}{2t_b} + \frac{Vt_f}{t_b} t - \frac{V}{2t_b} t^2 & t \in [t_f - t_b, t_f] \end{cases}$$

↑ Trapezoidal velocity profile trajectory

→ Minimum Time Trajectory :

↓ Fastest trajectory from q_0 to q_f

↓ with a given constant acceleration 'a'.

Bang-Bang Trajectory

↓

Accelerate with maximum value $+a$ till an appropriate switching time t_s is reached, and then reduce the acceleration to its minimum value $-a$.

↓

We consider the developments for trapezoidal trajectory & find switching time t_s .

Let us assume that the velocity at $t_s = V_s$.

We then have $V_s = a t_s$.

Let the time t_b in (1) for trapezoidal trajectory be t_s . Then from (1) we have

$$t_b = t_s = \frac{q_0 - q_f + V_s t_f}{V_s}$$

For symmetry, we consider that $t_s = \frac{t_f}{2}$
 $\Rightarrow t_f = 2t_s$

Then

$$\Rightarrow t_s = \frac{q_0 - q_f}{V_s} + 2t_s$$

$$\Rightarrow -t_s = \frac{q_0 - q_f}{V_s}$$

$$\Rightarrow V_s = \frac{q_f - q_0}{t_s}$$

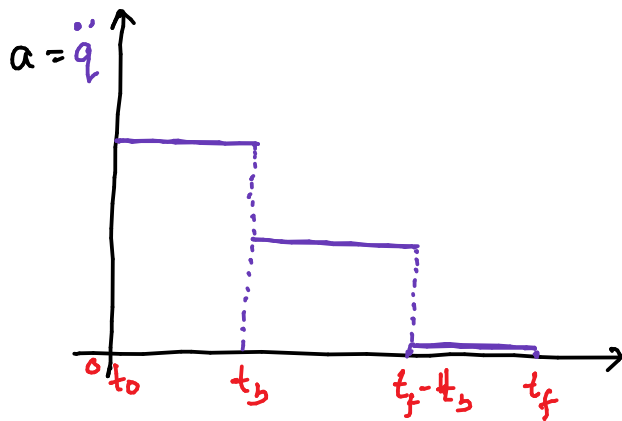
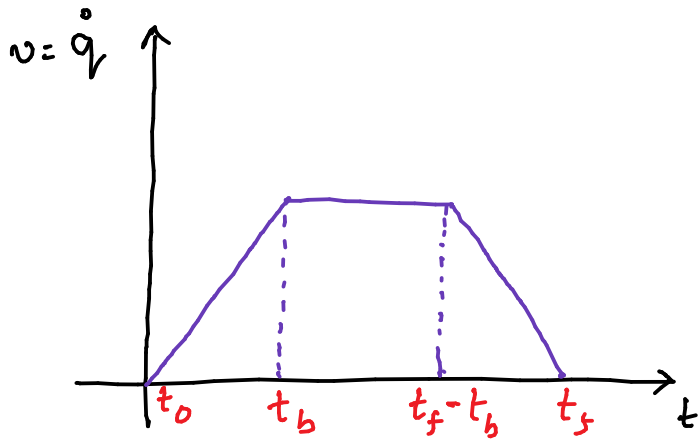
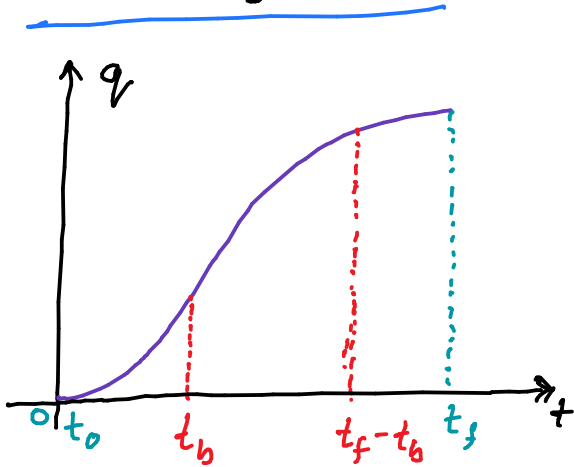
$$\Rightarrow a t_s = \frac{q_f - q_0}{t_s}$$

$$\Rightarrow t_s^2 = \frac{q_f - q_0}{a}$$

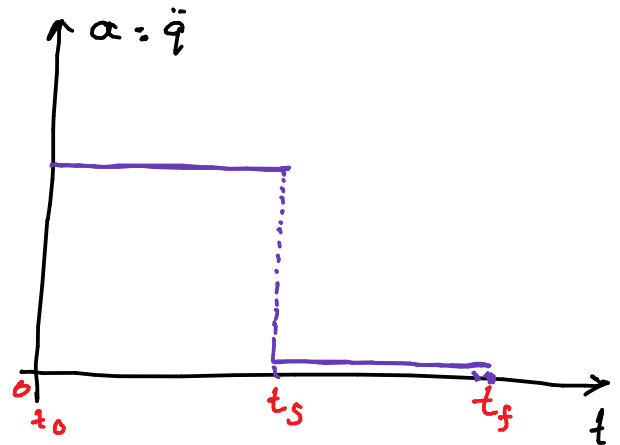
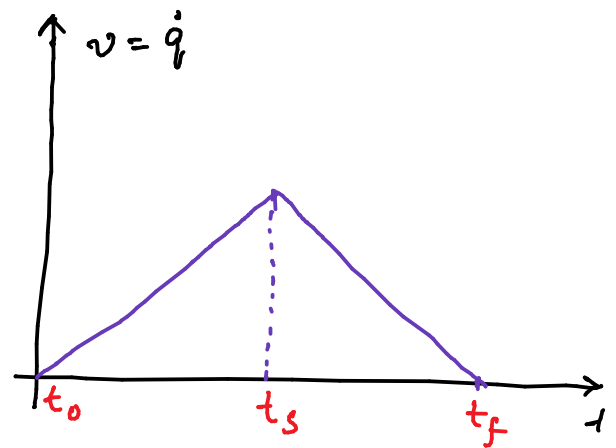
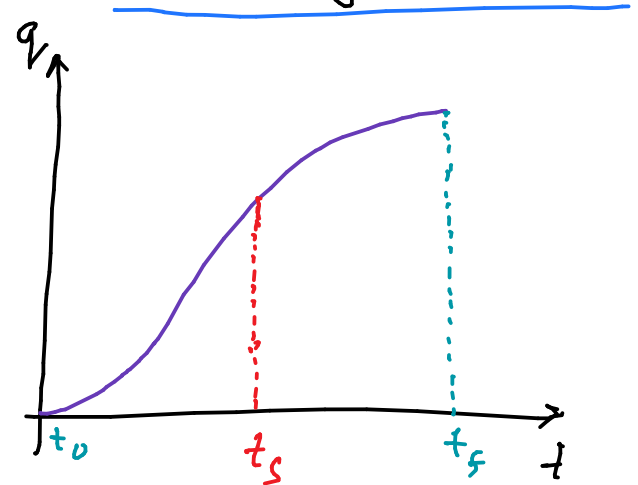
$$\Rightarrow t_s = \sqrt{\frac{q_f - q_0}{a}}$$

← switching time.

- For Trapezoidal Trajectory



- For minimum-time trajectory



→ Trajectories for Paths Specified by Via Points:

Consider a case where a path is specified by a sequence of configurations:

$$\{q_0, q_1, q_2, \dots\} \leftarrow \begin{matrix} \text{called} \\ \text{Via Points} \end{matrix}$$

Then, we need to plan for a trajectory that passes through a sequence of configurations i.e. through via points, satisfying some time-law.



In this case we can have a trajectory $q(t)$ with time instances $t_0, t_1, t_2, \dots$ s.t.

$$q(t_0) = q_0$$

$$q(t_1) = q_1$$

$$q(t_2) = q_2$$

$\vdots$

For $q(t)$ we can also impose velocity & acceleration constraints as well. For instance let a path has followed via points:

$$\{q_0, q_1, q_2\}$$

We have following constraints:

They can be satisfied by choosing a 6th deg. polynomial as trajectory

↓
i.e.

$$\begin{aligned} q(t_0) &= q_1 \\ q(t_1) &= q_2 \\ q(t_2) &= q_3 \\ \dot{q}(t_0) &= v_0 \\ \ddot{q}(t_0) &= a_0 \\ \dot{q}(t_2) &= v_2 \\ \ddot{q}(t_2) &= a_2 \end{aligned}$$

$$q(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_6 t^6$$

Another alternative approach, instead of using a single higher degree polynomial

Use low degree polynomials for trajectory segments between adjacent via points.

Referred to as: Interpolating Polynomials
or
Blending Polynomials

With this approach it is required to ensure that the velocity & acceleration constraints are satisfied at via points, where we switch from one low degree polynomial to another low degree polynomial.

↑ Read the details from the book
(Exercise)