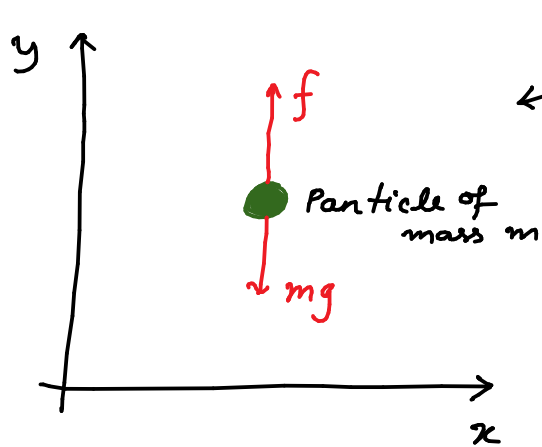


Part-1

Robot kinematics: Describe the motion of the robot without considering by which force and the motion has happened.

Robot dynamics: Establish an explicit relationship between force/torque and the motion of the robot.

→ The Euler-Lagrange Equation:



← One DOF system

By Newton's Second law, the equation of motion of the particle is :

$$m\ddot{y} = f - mg$$

- Further, note that :

$$\begin{aligned}
 m\ddot{y} &= \frac{d}{dt} (m\dot{y}) \\
 &= \frac{d}{dt} \frac{\partial}{\partial \dot{y}} \left(\frac{1}{2} m \dot{y}^2 \right) \\
 &= \frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{y}} \dots (*)
 \end{aligned}$$

where $\mathcal{K} = \frac{1}{2} m \dot{y}^2$ is the Kinetic energy associated with the particle.

Similarly we can express :

$$mg = \frac{\partial}{\partial y} (mgy) = \frac{\partial \mathcal{P}}{\partial y} \dots (**)$$

where $\mathcal{P} = mgy$ is the potential energy associated with the particle.

- Define the Lagrangian of the system as follows $\mathcal{L} = \mathcal{K} - \mathcal{P}$

Then we have

$$\mathcal{L} = \frac{1}{2} m \dot{y}^2 - mgy$$

↑
Energy function

Observe that

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{y}} &= \frac{\partial}{\partial \dot{y}} \left[\frac{1}{2} m \dot{y}^2 - mgy \right] \\ &= m\dot{y} = \frac{\partial \mathcal{K}}{\partial \dot{y}} \quad \dots \quad (1)\end{aligned}$$

$$\begin{aligned}2 \quad \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial}{\partial y} \left[\frac{1}{2} m \dot{y}^2 - mgy \right] \\ &= -mg \\ &= -\frac{\partial \mathcal{P}}{\partial y} \quad \dots \quad (2)\end{aligned}$$

Then using (1) & (2), the Newton's second law can be written as:

$$m\ddot{y} + mg = f$$

$$\frac{d}{dt}(m\dot{y}) + mg = f$$

$$\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{y}} + \frac{\partial \mathcal{P}}{\partial y} = f$$

$$\boxed{\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f}$$

↑ ↑ The Euler-Lagrange Equation

This is an alternative "energy-based" dynamical system representation.



Another one is Newton's second law based model:

$$m\ddot{y} + mg = f$$

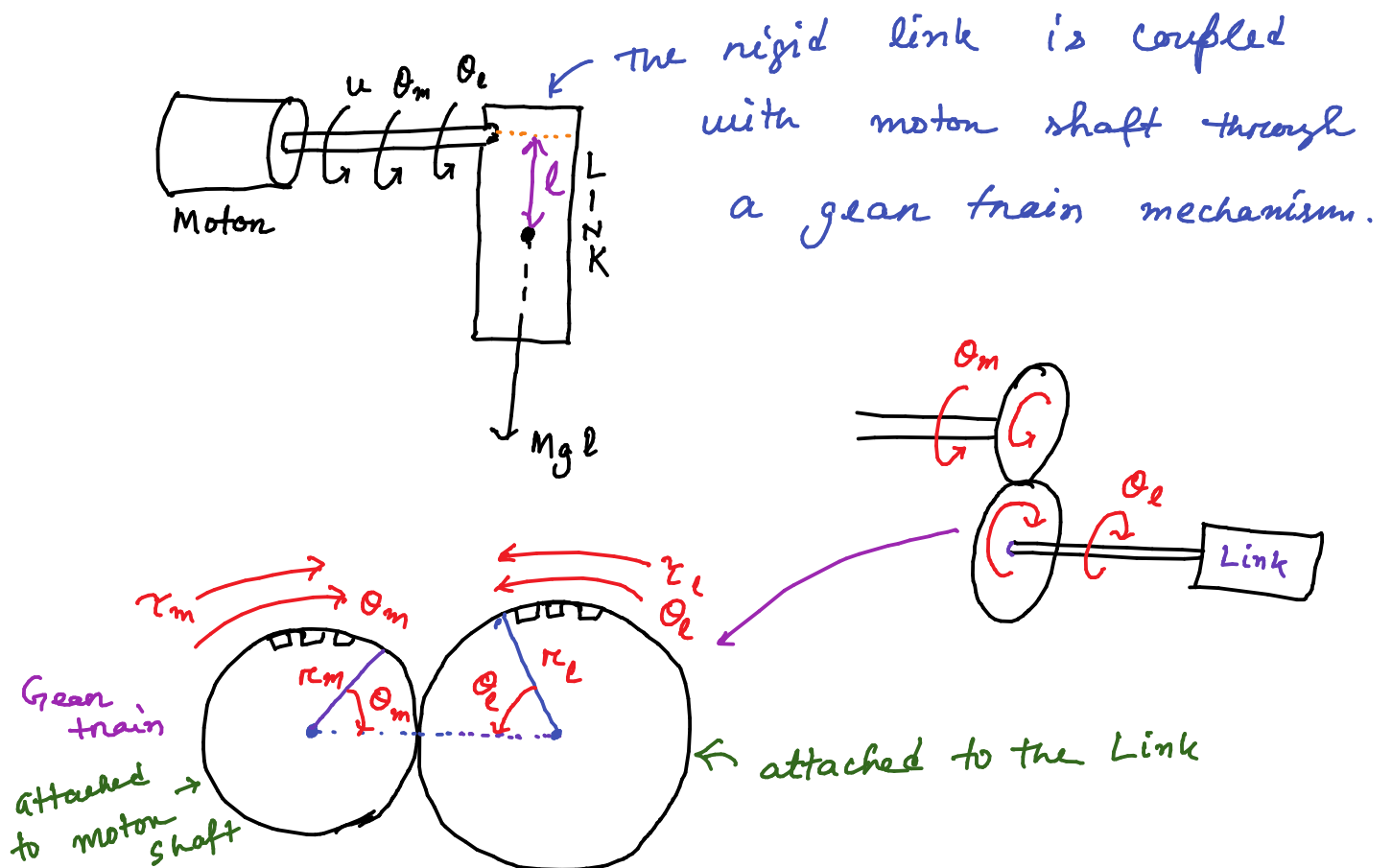
- For an 'n'-DOF system with co-ordinates $\{q_1, q_2 \dots q_n\}$, the Euler-Lagrange eqⁿ can be written as:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k \quad \text{for } k=1, 2 \dots n$$

where τ_k is the force/torque associated with q_k .

Example:

Consider a Single-Link robotic manipulator as in below fig.



Following conditions hold for gear-train:

$$\begin{aligned} \bullet \quad \frac{r_m}{r_l} &= \frac{N_m}{N_l} \\ \bullet \quad r_l \theta_l &= r_m \theta_m \\ \bullet \quad \tau_l \theta_l &= \tau_m \theta_m \end{aligned}$$

$$N_l \geq N_m$$

are the number of teeth in the gears.

τ_i : torques

From the above relatⁿ

$$\theta_m = \frac{r_l}{r_m} \theta_l = \frac{N_l}{N_m} \theta_l$$

↑
Gear ratio

let the gear ratio be $\pi:1$

$$\text{i.e.} \quad \frac{N_l}{N_m} = \frac{\pi}{1}$$

Typically for robotic arms π is:

20 to 200 or more.

⇓

The no. of teeth in the gear attached with link is 20 to 200 times greater than or equal to the no. of teeth in gear attached to the motor.

Hence,

$$\theta_m = r \theta_l$$

The kinetic energy of the system is:

$$K = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 \quad J_i : \text{inertia}$$

$$= \frac{1}{2} J_m r^2 \dot{\theta}_l^2 + \frac{1}{2} J_l \dot{\theta}_l^2$$

$$= \frac{1}{2} (r^2 J_m + J_l) \dot{\theta}_l^2 \quad \dots \quad (3)$$

The potential energy:

$$P = Mgl (1 - \cos \theta_l)$$

when M : total mass of the link

l : distance from joint axis to the link center of mass

Let us define the total inertia of

the system as $J = r^2 J_m + J_l$. Then

the Lagrangian:

$$L = K - P$$

$$= \frac{1}{2} J \dot{\theta}_l^2 - mgl (1 - \cos \theta_l)$$

Then the equation of motion is computed from the following Euler-Lagrange Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

For this example there is one variable θ_l (other could be θ_m , but it is related to θ_l)

Hence EL equt' :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_l} - \frac{\partial L}{\partial \theta_l} = \tau_l$$

$$\Rightarrow \boxed{J \ddot{\theta}_l + Mgl \sin \theta_l = \tau_l}$$

The generalized force/torque τ_l represents all the external forces & torques that are not derivable from potential functions.

For this example, the external forces/torques are :

(i) the input motor torque τ_m

(ii) the damping torques

$$B_m \dot{\theta}_m \text{ \& } B_l \dot{\theta}_l$$

For algebraic manipulation, all the forces/torque needs to be referred to the motor shaft side or the link (load) side.



We refer all quantities to the link side.

Let the motor torque τ_m referring to link side be τ_m^l . Then following the

relation
$$\theta_m \tau_m = \theta_l \tau_m^l$$

$$\Rightarrow \tau_m^l = \frac{\theta_m}{\theta_l} \tau_m$$

$$\Rightarrow \tau_m^l = \pi \tau_m$$

Similarly, let the damping coefficient B_m associated with motor side, referring to link side be B_m^l . Then

$$B_m \dot{\theta}_m = B_m^l \dot{\theta}_l$$

$$\Rightarrow B_m^l = \frac{\dot{\theta}_m}{\dot{\theta}_l} B_m$$

$$\Rightarrow B_m^l = \pi B_m$$

Then the link side torques are:

$$\tau_l + \tau_m^l = \tau_l + r \tau_m$$

$$\begin{aligned} B_m^l \dot{\theta}_l + B_l \dot{\theta}_l &= r B_m \dot{\theta}_l + B_l \dot{\theta}_l \\ &= \underbrace{(r B_m + B_l)}_B \dot{\theta}_l \end{aligned}$$

Net torque on link side

$$\tau_l = r \tau_m - B \dot{\theta}_l$$

Then referring to the Euler-Lagrange eqⁿ:

$$J \ddot{\theta}_l + Mgl \sin \theta_l = \tau_l$$

$$\Rightarrow J \ddot{\theta}_l + Mgl \sin \theta_l = r \tau_m - B \dot{\theta}_l$$

$$\Rightarrow J \ddot{\theta}_l + B \dot{\theta}_l + Mgl \sin \theta_l = u$$

where

$$J = r^2 J_m + J_l$$

$$B = r B_m + B_l$$

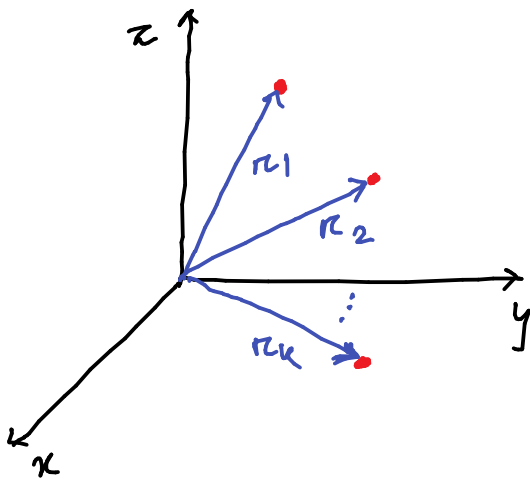
$$u = r \tau_m$$

Equation of motion of single link robot.

→ Multi-DOF System Motion:

↑
An extension of what we did for single DOF motion of a mass.

Consider a system of k -particles with their position vectors: $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$.

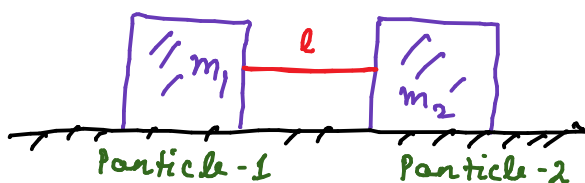


- When there is no restriction on motion of the above particles, then obtaining their equation of motion is easy i.e.

similar to what we had obtained for n -DOF system Euler-Lagrange eqⁿ:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k$$

- When there is restriction on motion of the particles due to some constrained such as in below fig., then it is



important to consider following forces

1. Externally applied force
2. Constraint Force

↑
force required to hold the constraints.

For instance consider a system of two rigid bodies which are attached through a mass-less wire of length l . Then the two coordinates r_1 & r_2 of the rigid bodies must satisfy

$$\|r_1 - r_2\| = l$$

$$\text{i.e. } (r_1 - r_2)^T (r_1 - r_2) = l^2$$



Hence, if external forces are applied to the particles, then they will experience these external forces + the constraint force that arise due to the connection of two particles through wire.

We develop a procedure to obtain equations of motions for such a constrained system.

- Holonomic Constraint

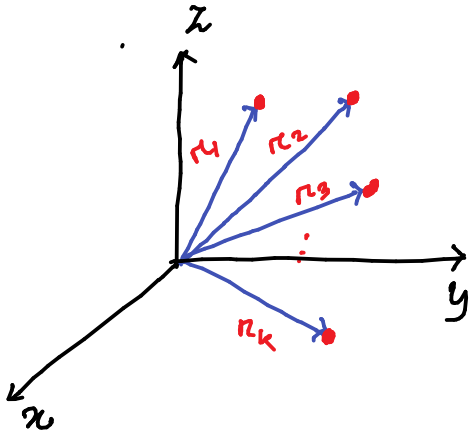
A constraint on the k -coordinates $r_1, r_2 \dots r_k$ is called holonomic if it is an equality algebraic equation:

$$g_i(r_1, r_2 \dots r_k) = 0 \quad i = 1, 2 \dots l$$

→ Generalized Coordinates:

Consider the k -particle system as before with their position vectors:

$$\{r_1, r_2, \dots, r_k\}$$



Note that each position vector can be written as 3-tuple of

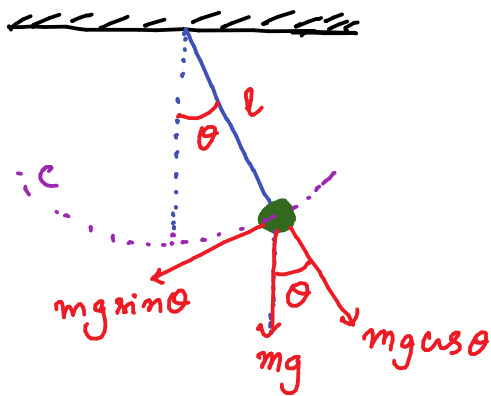
Cartesian coordinates:

$$r_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \quad \text{for } i=1, 2, \dots, k$$

With this Cartesian coordinate system representation, the k -particle system can be represented by $3k$ variables/quantities.

- When this k -particle system is constrained to satisfy some constraint equations, such as holonomic constraints, then it is possible to reduce some of the coordinates/variables to represent this system (instead of $3k$ coordinates).

- Consider the planar oscillation of a pendulum.



The position of the pendulum can be expressed using coordinates x & y i.e.

$$r = \begin{bmatrix} x \\ y \end{bmatrix}$$

Since the pendulum has to follow the path C , we have one constraint equation:

$$x^2 + y^2 - l^2 = 0$$

Considering the above constraint equation, we can reduce the number of co-ordinate variables as follows.

Let θ be the angular position of the pendulum from the vertical axis. Then, θ can be used to describe the coordinates x & y as :

$$x = l \sin \theta$$

$$y = -l \cos \theta$$

Then the position vector $r = \begin{bmatrix} l \sin \theta \\ -l \cos \theta \end{bmatrix}$.

With this new representation, we need only one coordinate variable Q to represent r .

- Due to the presence of one algebraic constraint equation, we could reduce one coordinate variable.

↓

Hence, it is expected that for a k -particle system with l - holonomic constraints, we can reduce the coordinate variables to : $n = 3k - l$ independent coordinate variables.

↓

These new variables are called "Generalized coordinate variables".

We denote them as:

$$\{q_1, q_2, \dots, q_n\}$$

Hence, the position vector of k -particle system can be represented as follows w.r.t. the (new) generalized coordinate variables:

$$r_i = r_i(q_1, q_2, \dots, q_n) \quad \text{for } i = 1, 2, \dots, k$$

- Now considering the two-particle system, joined by mass-less rigid wire, the constraint:

$$\boxed{(\mathbf{r}_1 - \mathbf{r}_2)^T (\mathbf{r}_1 - \mathbf{r}_2) = l^2 \quad \text{is a holonomic constraint}}$$

→ Consider the holonomic constraint equation:

$$g_i(r_1, r_2, \dots, r_k) = 0 \quad \text{for } i=1, 2, \dots, l$$

Then, by differentiating $g_i(\cdot)$, we have:

$$\boxed{\sum_{j=1}^k \frac{\partial g_i}{\partial r_j} \cdot dr_j = 0} \quad \dots \quad \textcircled{A}$$

• Nonholonomic Constraint:

A constraint of the form:

$$\boxed{\sum_{j=1}^k \omega_j \cdot dr_j = 0} \quad \text{Compare with } \textcircled{A}$$

is called nonholonomic if it can not be integrated to obtain an equality constraint of the form:

$$g_i(r_1, r_2, \dots, r_k) = 0.$$

→ Virtual Work :

Consider two particles with their position vectors : r_1 & r_2 , connected through a rigid bar of length l .

Assume that r_1 & r_2 are perturbed to $r_1 + \delta r_1$ & $r_2 + \delta r_2$.

↓

Since r_1 & r_2 satisfy



$$(r_1 - r_2)^T (r_1 - r_2) = l^2, \text{ the}$$

perturbed vectors must satisfy:

$$(r_1 + \delta r_1 - r_2 - \delta r_2)^T (r_1 + \delta r_1 - r_2 - \delta r_2) = l^2 \dots \textcircled{I}$$

verify $\left[\begin{array}{l} \text{By neglecting the quadratic terms in } \delta r_1 \text{ \& } \delta r_2, \text{ and using the fact } (r_1 - r_2)^T (r_1 - r_2) = l^2 \\ \text{it can be simplified } \textcircled{I} \text{ to:} \end{array} \right.$

$$(r_1 - r_2)^T (\delta r_1 - \delta r_2) = 0 \dots \textcircled{VD}$$

δr_1 & δr_2 are called
"virtual displacements".

- Let us represent each position vector r_i in the following form using n -generalized coordinates: $q_1, q_2 \dots q_n$

$$r_i = r_i(q_1, q_2, q_3, \dots q_n) \quad \dots \quad i=1, 2 \dots k \quad \textcircled{II}$$

where $q_1, q_2 \dots q_n$ are all independent variables.

With respect to the representation $\textcircled{II}$ of r_i , we can represent the virtual displacements as follows:

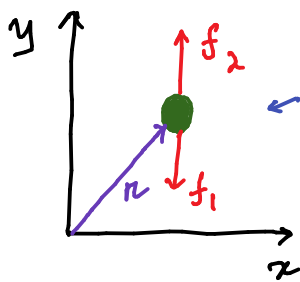
$$\delta r_i = \frac{\partial r_i}{\partial q_1} \delta q_1 + \frac{\partial r_i}{\partial q_2} \delta q_2 + \dots + \frac{\partial r_i}{\partial q_n} \delta q_n$$

where δq_i are associated with the generalized coordinates q_i .

- In the k -particles system, assume that each particle is in equilibrium.



The net force on each particle is zero.



← $f_1 + f_2 = 0$

By defining

Net force

$$F = f_1 + f_2,$$

we have

$$F = 0 \text{ at equilibrium}$$

The work done on a particle is:

$$W = F \cdot \delta r \quad (F^T \delta r)$$

↑
Small displacement



At equilibrium

$$W = F \cdot \delta r = 0 \cdot \delta r = 0$$

Since net force F_i on i^{th} particle is

zero, it implies the work done: $F_i^T \delta r_i$

by the virtual displacement δr_i is zero

$$\text{i.e. } F_i^T \delta r_i = 0$$



The sum of the work done by any set of virtual displacements is also zero i.e.

(W) ...

$$\sum_{i=1}^k F_i^T \delta r_i = 0$$

F_i is the net force on total force on particle i .

- Note that the net force acting on i^{th} particle F_i is consisting of :

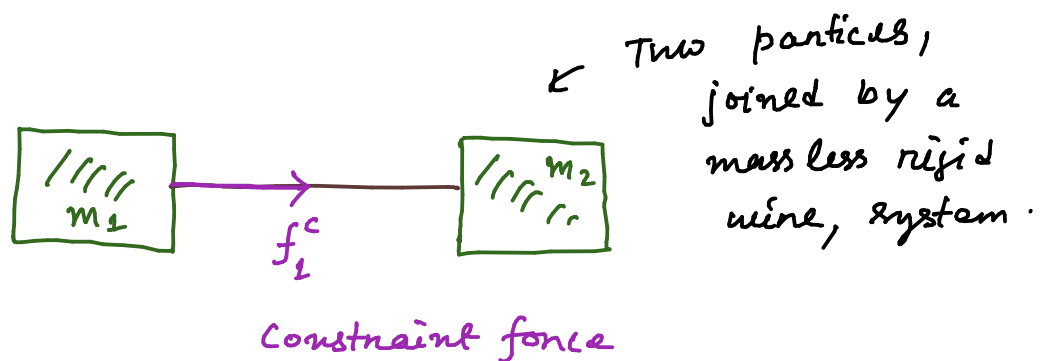
- (i), externally applied force f_i
- (ii), constraint force f_i^c

→ Assumption:

The total workdone by the constraint forces corresponding to any set of virtual displacements is zero, i.e.:

$$\sum_{i=1}^k (f_i^c)^T \delta r_i = 0 \quad \dots \quad \textcircled{A-1}$$

This will hold whenever the constraint force between a pair of particles is directed along the radial vector connecting the two particles.



f_1^c is directed along the radial vector connecting the two particles.

- Under the assumption (A-1), the sum of total work done by the virtual displacements, as in (W), becomes:

$$\sum_{i=1}^k F_i^T \delta r_i = 0$$

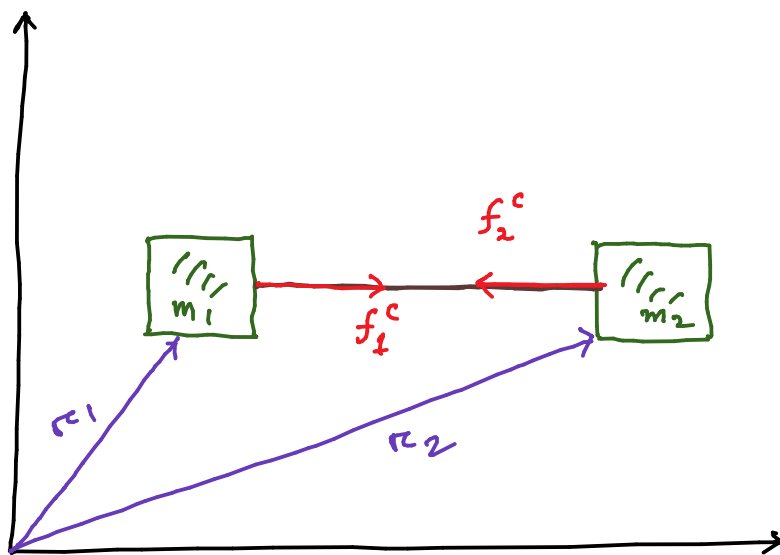
$$\Rightarrow \sum_{i=1}^k (f_i^T + f_i^c{}^T) \delta r_i = 0$$

$$\Rightarrow \sum_{i=1}^k f_i^T \delta r_i = 0$$

← "Principle of
Virtual
Work"

✓
The work done by external forces corresponding to any set of virtual displacement is zero, under assumption (A-1).

If the principle of virtual work applies for a system, then one can analyze the system on the work done by virtual displacements without considering the constraint forces.



- The constraint force exerted on particle 1 by the wire is of the following form:

$$f_1^c = \gamma (r_1 - r_2) \quad \gamma \text{ is some constant}$$

- The constraint force exerted on particle 2 by wire is of -ve of f_1^c :

$$f_2^c = -\gamma (r_1 - r_2)$$

- Now the work done by the constraint forces corresponding to a set of virtual displacements is:

$$(f_1^c)^T \delta r_1 + (f_2^c)^T \delta r_2 = \gamma \underbrace{[(r_1 - r_2)^T (\delta r_1 - \delta r_2)]}_{=0}$$

(see eqn (VD) before)

Hence, the considered two particles system satisfies the principle of virtual work.

- Hence, the Principle of Virtual Work applies to all systems which consists of several particles that are pairwise connected by rigid massless wires of fixed lengths.

↑

For robotic manipulation the principle of virtual work applies.