

ELL 406

ROBOT DYNAMICS

Part-2

In stating the Principle of Virtual Work, we had assumed that in the k -particle system, each particle is in Equilibrium, that is, the net force F_i on each particle is zero.



The sum of work done by any set of virtual displacements is zero. i.e.

$$\boxed{\sum_{i=1}^k F_i^T \delta r_i = 0} \dots \dots \textcircled{VW}$$

→ k -particle system where particles are Not in Equilibrium :

↑ Use D'Alembert's Principle

When the particles are not in equilibrium, the D'Alembert's Principle can be applied & modify the force equation to make each particle in equilibrium.

D'Alembert's Principle

Consider a k -particle system where each particle is not in equilibrium. For such systems, D'Alembert's Principle states that:

"Each particle will be in equilibrium, if one introduces an additional fictitious force: $-\dot{p}_i$, where p_i the momentum of i th particle.



By modifying F_i to $F_i - \dot{p}_i$, the equation in (VW) will remain same, i.e.

$$\sum_{i=1}^k (F_i - \dot{p}_i)^T \delta r_i = 0$$



We can also neglect the constraint forces, and apply the Principle of Virtual work



$$\sum_{i=1}^k f_i^T \delta r_i - \sum_{i=1}^k \dot{p}_i^T \delta r_i = 0 \quad \dots \dots \text{(VW-NE)}$$

- Now we will see how equation $\overline{VW-NE}$ can be represented in terms of the generalized coordinates, which are independent.

Since $\pi_i = \pi_i(q_1, q_2, \dots, q_n)$,

We have: $\delta \pi_i = \sum_{j=1}^n \frac{\partial \pi_i}{\partial q_j} \delta q_j$ for $i = 1, 2, \dots, k$

$\Downarrow$ The virtual work done by f_i 's:

$$\sum_{i=1}^k f_i^T \delta \pi_i = \sum_{i=1}^k f_i^T \left(\sum_{j=1}^n \frac{\partial \pi_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{i=1}^k \sum_{j=1}^n f_i^T \frac{\partial \pi_i}{\partial q_j} \delta q_j$$

$$= \sum_{j=1}^n \psi_j \delta q_j$$

where

$$\psi_j = \sum_{i=1}^k f_i^T \frac{\partial \pi_i}{\partial q_j}$$

$\uparrow$ called j^{th} Generalized force".

- Considering the 2nd term of VW-NE :

$$\sum_{i=1}^k \dot{p}_i^T \delta \pi_i = \sum_{i=1}^k m_i \ddot{r}_i^T \delta r_i \quad \left(\begin{array}{l} \text{momentum } p = mv \\ = m\dot{r} \end{array} \right)$$

$$= \sum_{i=1}^k m_i \ddot{r}_i^T \left(\sum_{j=1}^n \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{i=1}^k \sum_{j=1}^n m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

We have the following using chain rule:

$$\frac{d}{dt} \left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] = m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i^T \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right]$$

$$\Rightarrow m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt} \left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right]$$

Doing summation over all $i=1 \dots n$, we have:

$$\sum_{i=1}^k m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^k \left[\frac{d}{dt} \left(m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i^T \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right]$$

..... (M)

Further, since $r_i = r_i(q_1, q_2, \dots, q_n)$, we have:

$$\text{velocity : } v_i = \frac{dr_i}{dt} = \sum_{j=1}^n \frac{\partial r_i}{\partial q_j} \frac{dq_j}{dt}$$

$$v_i = \frac{d\pi_i}{dt} = \frac{\partial \pi_i}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \pi_i}{\partial q_2} \frac{dq_2}{dt} + \dots + \frac{\partial \pi_i}{\partial q_n} \frac{dq_n}{dt}$$

$$= \frac{\partial \pi_i}{\partial q_1} \dot{q}_1 + \frac{\partial \pi_i}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial \pi_i}{\partial q_n} \dot{q}_n \dots (*)$$



$$\boxed{\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial \pi_i}{\partial q_j}} \dots \textcircled{1}$$

- Using the "interchange property" of partial derivatives & ordinary differentiation, we have:

$$\boxed{\frac{d}{dt} \left[\frac{\partial \pi_i}{\partial q_j} \right] = \frac{\partial}{\partial q_j} \left[\frac{d\pi_i}{dt} \right]}$$

$$= \frac{\partial v_i}{\partial q_j} \quad \dots \textcircled{2}$$

(Using $\textcircled{*}$)

→ Now using (1) & (2), and the fact $v_i = \dot{r}_i$, we have the following from previous relation:

$$\sum_{i=1}^k m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^k \left[\frac{d}{dt} \left(m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i^T \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right]$$



$$\sum_{i=1}^k m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^k \left[\frac{d}{dt} \left(m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right]$$

∴ ... (EM)

By defining the Kinetic Energy to be as:

$$K = \sum_{i=1}^k \frac{1}{2} m_i v_i^T v_i$$

the relatⁿ (EM) can compactly be written as:

(Verify this)

$$\sum_{i=1}^k m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \quad \dots \text{(EM1)}$$

• Recall that

$$\begin{aligned}\sum_{i=1}^k \dot{\mathbf{p}}_i^T \delta \mathbf{r}_i &= \sum_{i=1}^k \sum_{j=1}^n m_i \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \delta \mathbf{q}_j \\ &= \sum_{j=1}^n \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{\mathbf{q}}_j} - \frac{\partial \mathcal{K}}{\partial \mathbf{q}_j} \right] \delta \mathbf{q}_j \quad \dots \quad \textcircled{A} \\ &\quad \text{(Using EM1)}\end{aligned}$$

Further, recall:

$$\sum_{i=1}^k \mathbf{f}_i^T \delta \mathbf{r}_i = \sum_{j=1}^n \psi_j \delta \mathbf{q}_j \quad \dots \quad \textcircled{A1}$$

Then, using $\textcircled{A}$ & $\textcircled{A1}$ in the total work done relation:

$$\begin{aligned}\sum_{i=1}^k \mathbf{f}_i^T \delta \mathbf{r}_i - \sum_{i=1}^k \dot{\mathbf{p}}_i^T \delta \mathbf{r}_i &= 0 \\ \Rightarrow \sum_{j=1}^n \psi_j \delta \mathbf{q}_j - \left[\sum_{j=1}^n \left(\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{\mathbf{q}}_j} - \frac{\partial \mathcal{K}}{\partial \mathbf{q}_j} \right) \delta \mathbf{q}_j \right] &= 0\end{aligned}$$

$$\Rightarrow \sum_{j=1}^n \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{\mathbf{q}}_j} - \frac{\partial \mathcal{K}}{\partial \mathbf{q}_j} - \psi_j \right] \delta \mathbf{q}_j = 0$$

VW-NE1

Now, since δq_j are independent, each coefficient of equation VW-NEI are zero, i.e. :

$$\boxed{\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j} \quad \text{for } j=1, 2, \dots, n$$

..... ELO

• If the generalized force ψ_j is the sum of :

- (i) externally applied generalized force χ_j , and
- (ii) potential field : $-\frac{\partial P}{\partial q_j}$

where $P(q)$ is the Potential Energy function, i.e.-

$$\psi_j = -\frac{\partial P}{\partial q_j} + \chi_j, \quad \text{then } \text{ELO} \text{ can be expressed as:}$$

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = -\frac{\partial P}{\partial q_j} + \chi_j$$

$$\Rightarrow \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} + \frac{\partial P}{\partial q_j} = \tau_j$$

Then, defining the Lagrangian function

$$\underline{L = K - P}, \text{ we have}$$

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \tau_j}$$

↑ Reached to .

"Euler-Lagrange Equation" of
motion

for constrained
systems.

- From the above Euler-Lagrange equation, it is very straightforward to obtain equation of motion or model of a dynamical system. However, for this one needs to compute the kinetic energy & potential energy of a rigid body using the independent generalized coordinate variables.

- R can be considered as the rotational transform matrix required for transform' with the body attached frame and inertial frame.
- The inertial tensor matrix is computed as follows:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where

$$I_{xx} = \iiint (y^2 + z^2) \rho(x, y, z) dx dy dz$$

$$I_{yy} = \iiint (x^2 + z^2) \rho(x, y, z) dx dy dz$$

$$I_{zz} = \iiint (x^2 + y^2) \rho(x, y, z) dx dy dz$$

$$I_{xy} = I_{yx} = - \iiint xy \rho(x, y, z) dx dy dz$$

$$I_{xz} = I_{zx} = - \iiint xz \rho(x, y, z) dx dy dz$$

$$I_{yz} = I_{zy} = - \iiint yz \rho(x, y, z) dx dy dz$$

where $\rho(x, y, z)$: mass density of the object represented as a funtⁿ of position

The integrals are computed over the region occupied by the body.

For a n -link robotic manipulator, the joint variables q_i 's are the generalized coordinate variables.

- Recall from the velocity kinematics that :

$$v_i = J_{v_i} \dot{q} \quad \& \quad \omega_i = J_{\omega_i} \dot{q}$$

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \text{ is the Jacobian matrix.}$$

- The overall K.E. of the manipulator is:

$$K = \frac{1}{2} \dot{q}^T \underbrace{\left[\sum_{i=1}^n (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I_i R_i^T J_{\omega_i}) \right]}_{D(q)} \dot{q}$$

The inertia matrix is symmetric & positive definite.

$D(q)$ is also symmetric positive definite matrix

$D(q)$ is zero if and only if all the joint velocities are zero.

→ Potential Energy for an n -link Manipulator:

- Assume that the mass of the entire object of an i^{th} -link (link + joint) is concentrated at its center of mass. Then the potential energy of i^{th} link:

$$\mathcal{P}_i = m_i g^T r_{ci}$$

Where g : is the vector giving the direction of gravity in the inertial frame

r_{ci} : is the vector that gives the coordinates of the center of mass of i^{th} link.

→ Then the total potential energy of the manipulator:

$$\mathcal{P} = \sum_{i=1}^n \mathcal{P}_i = \sum_{i=1}^n m_i g^T r_{ci}$$

→ Equation of Motion of n-link Manipulator :

- We obtained

$$K.E : K = \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

↓ can be represented as

$$= \sum_{ij} d_{ij}(q) \dot{q}_i \dot{q}_j$$

where d_{ij} are the entries of matrix $D(q)$.

Note that $P = P(q)$ is independent of $\dot{q}$.

- Then the Lagrangian energy function :

$$L = K - P$$

$$= \frac{1}{2} \sum_{ij} d_{ij}(q) \dot{q}_i \dot{q}_j - P(q)$$



The partial derivative of L w.r.t. k^{th} joint velocity :

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \dot{q}_j$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \ddot{q}_j + \sum_j \frac{d}{dt} d_{kj} \dot{q}_j$$

$$= \sum_j d_{kj} \ddot{q}_j + \sum_{ij} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$

Similarly

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{ij} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial \mathcal{P}}{\partial q_k}$$

Hence, for each $k = 1, 2, \dots, n$, the E-L equation:

$$\sum_j d_{kj} \ddot{q}_j + \sum_{ij} \left[\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{kj}}{\partial q_k} \right] \dot{q}_i \dot{q}_j + \frac{\partial \mathcal{P}}{\partial q_k} = \tau_k$$

Show this!

$$\sum_{ij} \left[\frac{\partial d_{kj}}{\partial q_i} \right] \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{ij} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i \dot{q}_j$$

Hence:

$$\sum_{ij} \left[\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \dot{q}_j = \sum_{ij} \frac{1}{2} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \dot{q}_j$$

$$= \sum_{ij} c_{ijk} \dot{q}_i \dot{q}_j$$

$$(c_{ijk} = c_{jik})$$

called Christoffel symbols (of first kind)

Define $g_k = \frac{\partial \mathcal{P}}{\partial q_k}$

then the EL equation is:

(D1)
$$\sum_{j=1}^n d_{kj}(q) \ddot{q}_j + \sum_{i=1}^n \sum_{j=1}^n c_{ijk}(q) \dot{q}_i \dot{q}_j + g_k(q) = \tau_k, \quad k = 1, 2, \dots, n$$

↓ Can be compactly represented as:

$$D(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) = \tau$$

where $(k, j)^{th}$ element of $C(q, \dot{q})$ is:

$$\begin{aligned} c_{kj} &= \sum_{i=1}^n c_{ijk}(q) \dot{q}_i \\ &= \sum_{i=1}^n \frac{1}{2} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \end{aligned}$$

and gravity vector

$$g(q) = [g_1(q) \quad g_2(q) \quad \dots \quad g_n(q)]^T$$

→ See the Examples in Spong, Hutchinson & Vidyasagar

Another way of modeling a robot is by "Newton-Euler" formulation.

book for derivation of EL equation of motion for different manipulators.

• In the relation (1), the terms involving

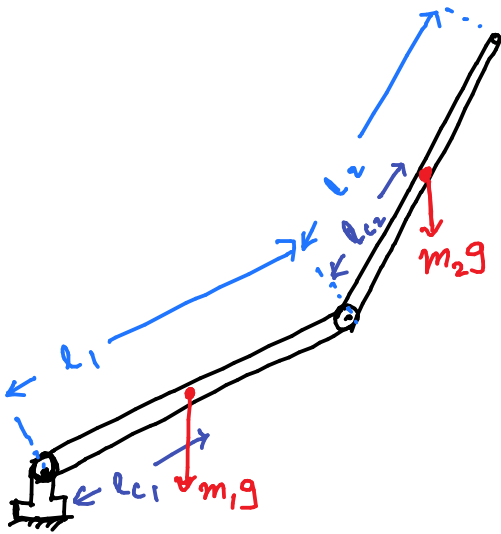
$\dot{q}^2$ are called "centrifugal" &

$\dot{q}_i \dot{q}_j$ are called "Coriolis" terms.

↓

For instance, for a two-link planar manipulator the equations of motions are:

$$\begin{cases} d_{11} \ddot{q}_1 + d_{12} \ddot{q}_2 + c_{121} \dot{q}_1 \dot{q}_2 + \underbrace{c_{211} \dot{q}_2 \dot{q}_1}_{\text{Coriolis term}} + \underbrace{c_{221} \dot{q}_2^2}_{\text{centrifugal term}} + g_1 = \tau_1 \\ d_{21} \ddot{q}_1 + d_{22} \ddot{q}_2 + c_{112} \dot{q}_1^2 + g_2 = \tau_2 \end{cases}$$



$$d_{11} = m_1 l_{c1}^2 + m_2 (l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos q_2) + I_1 + I_2$$

$$d_{12} = d_{21} = m_2 (l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_2$$

$$d_{22} = m_2 l_{c2}^2 + I_2$$

• The Christoffel symbols:

$$c_{111} = \frac{1}{2} \frac{\partial d_{11}}{\partial q_1} = 0$$

$$c_{121} = c_{211} = \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = - \underbrace{m_2 l_1 l_{c2} \sin q_2}_{\substack{h \\ h}}$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = h$$

$$C_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = -l$$

$$g_1 = \frac{\partial P}{\partial q_1}$$

$$C_{122} = C_{212} = \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = 0$$

$$g_2 = \frac{\partial P}{\partial q_2}$$

$$C_{222} = \frac{1}{2} \frac{\partial d_{22}}{\partial q_2} = 0$$

→ Another way to get robot dynamics is
by "Newton-Euler Formulation".

(Skipping from this course)