

ROBOT CONTROLPart - 1

- Robot Control: The problem of determining the appropriate controlled force/torque to the joints of a manipulator to execute so desired task by the end-effector.



One of the simplest robot control procedure is "Independent Joint Control" method.

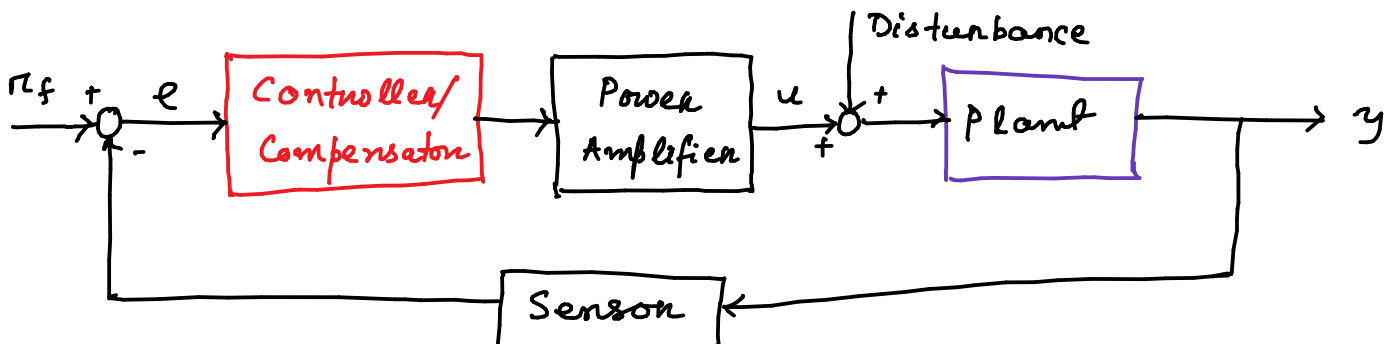


Each axis/link of a manipulator is controlled as a single-input single-output system.



Ref. trajectory

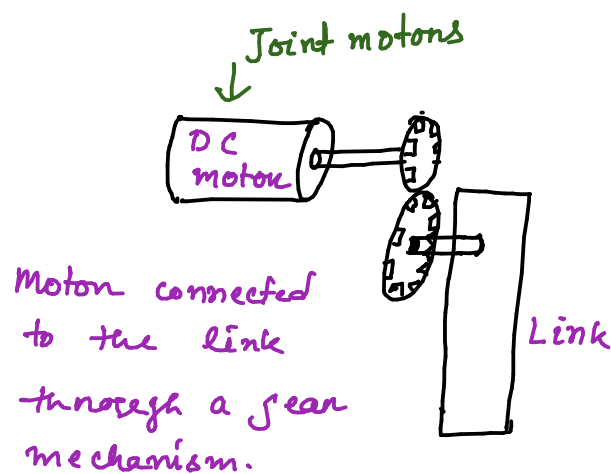
Standard feedback arrangement



- The objective is to design a controller such that the plant output tracks the given reference trajectory. In addition, the effect of disturbance also needs to be suppressed/rejected. ↑

Reference tracking + Disturbance rejection

- Once an appropriate controller is designed, the control signal needs to be given to an actuator. In a manipulator, each link is connected with a joint whose motion is governed by the motors.

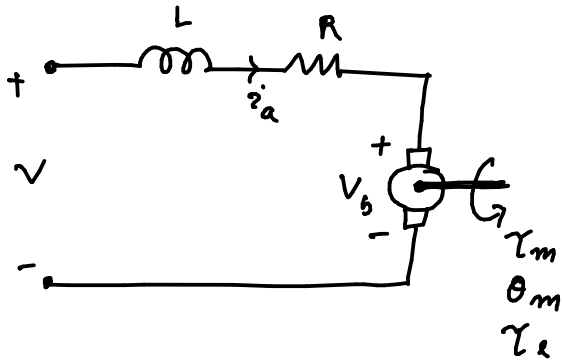


- Hence, the actuator for a particular link is the attached joint motor.

→ Dynamics of Joint Motors:

Joint motors are usually DC motors.

DC motor schematic Diagram



The differential eqⁿ for armature current:

$$L \frac{di_a}{dt} + R i_a = V - V_b$$

- The torque developed by motor:

$$\begin{aligned} \tau_m &= K_t \Phi i_a \\ &= K_m i_a \end{aligned}$$

For a permanent magnet DC motor field flux Φ is constant.

- The back emf developed:

$$\begin{aligned} V_b &= K_b \Phi \omega_m \\ &= K_b \frac{d\theta_m}{dt} \end{aligned}$$

ω_m : angular velocity of motor.

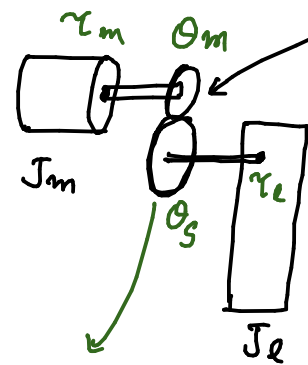
Usually K_m & K_b are same, & they can be determined from torque-speed characteristics.

$$K_m = \frac{R \tau_0}{V_r}$$

$$V_r = R i_a$$

τ_0 : blocked-rotor torque at the rated voltage.

→ Combined Model of Joint & Link (Plant)



Gear train with gear ratio $r : 1$.

we have following relation:

$$J_m \frac{d^2 \theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} = \tau_m - \tau_l / r$$

Now writing all differential equation in 's' domain (Laplace domain) :

$$(Ls + R) I_a(s) = V(s) - K_b s \theta_m(s)$$

$$(J_m s^2 + B_m s) \theta_m(s) = K_m I_a(s) - \frac{\tau_l(s)}{r}$$

$$J_m = J_a + J_g$$

J_a : Actuator inertia

J_g : Gear inertia

J_l : Link inertia

B_m : coefficient of motor friction (damping)

The overall transfer functions are :

- With $\tau_e = 0$

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s[(Ls+R)(J_ms+B_m) + K_b K_m]}$$

- With $V = 0$

$$\frac{\Theta_m(s)}{\tau_e(s)} = \frac{-(Ls+R)/r}{s[(Ls+R)(J_ms+B_m) + K_b K_m]}$$

- It is often assumed that the electrical time constant : L/R is much smaller than mechanical time constant : J_m/B_m . This assumption leads to approximated reduced order transfer functions:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s(J_ms+B_m + K_m K_b/R)}$$
$$\frac{\Theta_m(s)}{\tau_e(s)} = \frac{-1/r}{s(J_ms+B_m + K_m K_b/R)}$$

For the previous reduced order tfs,
we have following eqⁿ in time-domain

$$\begin{cases} J_m \ddot{\theta}_m + \left(B_m + \frac{K_b K_m}{R} \right) \dot{\theta}_m = \frac{K_m}{R} V(t) \\ 2 J_m \ddot{\theta}_m + \left(B_m + \frac{K_b K_m}{R} \right) \dot{\theta}_m = -\frac{1}{n} \tau_L(t) \end{cases}$$

Further,

For input $V(s)$, the output

$$\theta_m(s) = G(s) \cdot \frac{K_m}{R} \cdot V(s) \quad \dots \quad (1)$$

Similarly, for input $\tau_L(s)$, the output

$$\theta_m(s) = G(s) \cdot \left(-\frac{1}{n} \right) \cdot \tau_L(s) \quad \dots \quad (2)$$

Adding (1) & (2), we have:

$$\underbrace{G(s)}_{\substack{\mathcal{L} \\ \text{operation}}} \left[\underbrace{\frac{K_m}{R} V(s)}_{u_1} + \underbrace{\left(-\frac{1}{n} \right) \tau_L(s)}_{u_2} \right] = \underbrace{2 \theta_m(s)}_{y_1 + y_2}$$

↓

$$\mathcal{L}(u_1 + u_2) = \mathcal{L}(u_1) + \mathcal{L}(u_2)$$

Principle of superposition

Hence, by considering $\frac{K_m}{R} V(s)$ & $-\frac{1}{n} \chi_c(s)$ as inputs to the system, & applying the "principle of superposition", we can write the following 2nd order differential eqⁿ for the plant (joint + link) :

$$(*) \dots J \ddot{\theta} + B \dot{\theta} = u(t) + d(t)$$

u : will be considered
as control input
to the motor

$$J = J_m$$

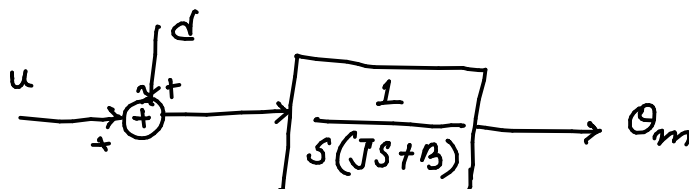
$$B = B_m + \frac{K_b K_m}{R}$$

$$u = \frac{K_m}{R} V$$

$$d = -\frac{1}{n} \chi_c$$

d : will be considered
as disturbance input that is
due to the coupling effects from
other links.

The relation (*) can be represented as :

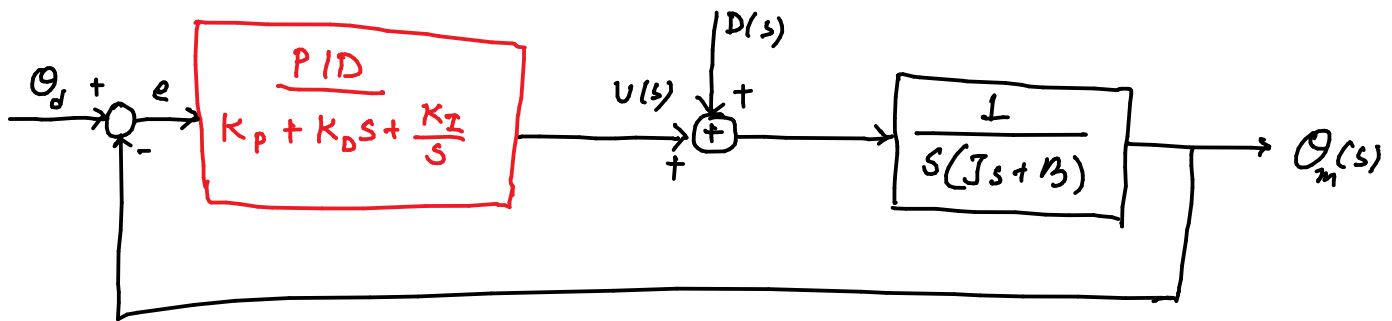


- Note that the disturbance $d \propto \frac{1}{n}$ i.e.
proportional to gear reduction $\frac{1}{n}$ factor.

↳ The objective is to track a 'constant' or 'step-reference' command Θ_d .

Common θ_d .

Month	Very satisfied (%)
March 2020	56
April 2020	60
May 2020	58
June 2020	57
July 2020	56
August 2020	55
September 2020	54
October 2020	53
November 2020	52
December 2020	51
January 2021	50
February 2021	49
March 2021	44



We use a PID controller $K(s)$, whose t.f. is:

$$K(s) = K_p + K_d s + \frac{K_I}{s}$$

$$K(s) = K_p + K_D s + \frac{K_I}{s}$$

where $K_P, K_D \geq K_I$ are design parameters -

• The control input is:

$$U(s) = K(s) [Q_d(s) - Q_m(s)]$$

• The system has two inputs: $\Theta_d(s)$ & $D(s)$

$\downarrow$
Reference

$\uparrow$
Disturbance

- By superposition, the output of the system $\Theta_m = y_1 + y_2$

L : Linear operator,

when $y_1 = L(u_1)$
 $y_2 = L(u_2)$



$$\Theta_m(s) = \frac{(K_D s^2 + K_P s + K_I)}{\sigma(s)} \Theta_d(s) - \frac{r s}{\sigma(s)} D(s)$$

where $\sigma(s) = J s^3 + (B + K_D) s^2 + K_P s + K_I$

- Applying Routh-Hurwitz Criterion

$$\left. \begin{array}{l} K_P > 0, K_D > 0, K_I > 0 \\ \& K_I < \frac{(B + K_D) K_P}{J} \end{array} \right\} \Rightarrow \text{The closed loop is stable.}$$

- The Steady State Error is:

$$\lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} (\Theta_d(s) - \Theta_m(s)) s$$

The disturbance can be considered as constant disturbance i.e.

$$D(s) = \frac{D}{s} \quad D: \text{magnitude of const signal.}$$

- Since the reference Θ_d is a step signal,

we have : $\Theta_d(s) = \frac{R_f}{s}$

where R_f is magnitude of step signal.

• Hence,

$$\lim_{s \rightarrow 0} s (\Theta_d(s) - \Theta_m(s)) = \lim_{s \rightarrow 0} s \left(1 - \frac{K_D s^2 + K_P s + K_I}{\sigma(s)} \right) \Theta_d(s) + \frac{\tau s}{\sigma(s)} \cdot \frac{D}{s}$$

$$= \lim_{s \rightarrow 0} s \left[\left(\frac{\sigma(s) - K_D s^2 + K_P s + K_I}{\sigma(s)} \right) \frac{R_f}{s} + \frac{\tau D}{\sigma(s)} \right]$$

$$= \lim_{s \rightarrow 0} \left[\frac{s (J s^2 + B s)}{\sigma(s)} \cdot R_f + \frac{\tau D s}{\sigma(s)} \right]$$

$$= 0$$

⇓

The compensator will track the given constant ref. signal Θ_d .

* For design of PID controller $\rightarrow$ Refer to

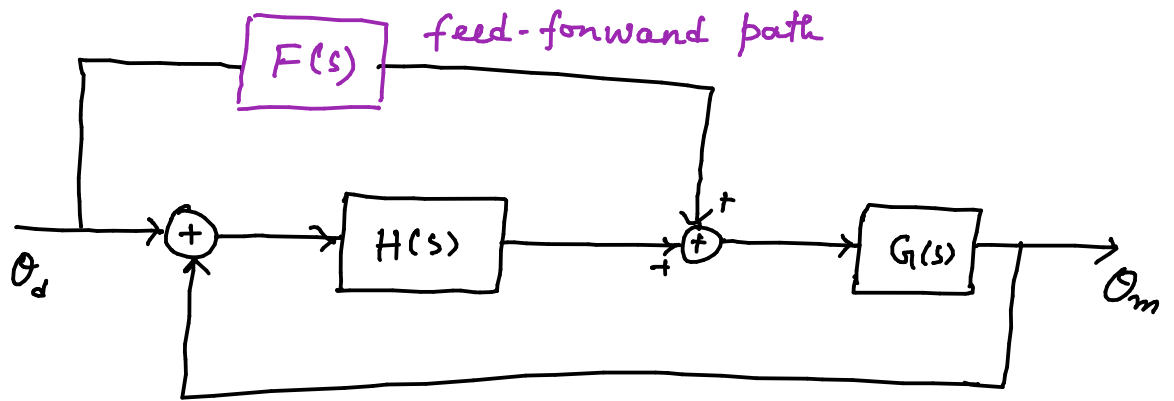
ELL 225

lecture notes.

$\rightarrow$ Tracking of Time-varying Reference Trajectory:

- Assume that the external reference Θ_d is now an arbitrary joint-space reference trajectory.

→ We use a "Feed-forward Control" scheme to address trajectory tracking problem.



Let $G(s) = \frac{q(s)}{p(s)}$ (Plant) $H(s) = \frac{c(s)}{d(s)}$ (controller)

$F(s) = \frac{a(s)}{b(s)}$ (Feedforward t.f.)

Let $G(s)$ be strictly proper

$\deg(\text{num}) < \deg(\text{den})$

• The closed-loop t.f. is :

$$T(s) = \frac{q(s) [c(s)b(s) + a(s)d(s)]}{b(s) [p(s)d(s) + q(s)c(s)]}$$

- The closed-loop characteristic polynomial:

$$\sigma(s) = b(s) [p(s)d(s) + q(s)c(s)]$$



For stability of closed loop system, the controller $H(s)$ & feedforward t.f. $F(s)$ must be designed s.t. the roots of $\sigma(s)$ be on open left half of complex plane.



Also observe the feedforward t.f. is also stable.

→ Assume that the numerator polynomial of $G(s)$ i.e. $q(s)$ is stable (roots are in open left half of complex plane)



Choose $F(s) = \frac{1}{G(s)}$

i.e. $a(s) = p(s)$ & $b(s) = q(s)$

With this, the closed-loop t.f. $\frac{\Theta_m(s)}{\Theta_d(s)}$:

$$T(s) = \frac{q(s) [c(s)b(s) + a(s)d(s)]}{b(s) [p(s)d(s) + q(s)c(s)]}$$

$$= \frac{q(s) [c(s)q(s) + p(s)d(s)]}{q(s) [p(s)d(s) + c(s)q(s)]}$$

$$\Downarrow$$
$$\Theta_m(s) = \frac{q(s) [c(s)q(s) + p(s)d(s)]}{q(s) [p(s)d(s) + c(s)q(s)]} \Theta_d(s)$$

The error

$$E(s) = \Theta_m(s) - \Theta_d(s)$$
$$= \left[\frac{q(s) [c(s)q(s) + p(s)d(s)]}{q(s) [p(s)d(s) + c(s)q(s)]} - 1 \right] \Theta_d(s)$$

$$\Rightarrow E(s) = 0$$

$\Downarrow$

The closed-loop system will track any desired trajectory Θ_d provided the closed-loop system is stable.

→ Applying this to the robot model:

$$G(s) = \frac{1}{s(Js+B)} \Rightarrow F(s) = Js^2 + Bs$$

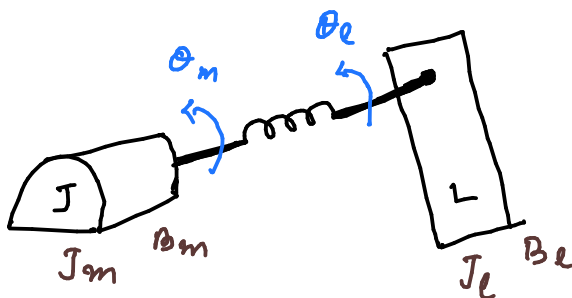
$$\text{Let } H(s) = K_p + K_d s \quad (\text{PD-compensator})$$

Note that $1/G(s)$ is not a Proper transfer function. However, since $\theta_d(t)$ is a smooth time-varying trajectory, its derivative can be pre-computed, & then pass through the forward path block.

$$\Downarrow \left[Js^2 \theta_d(s) \rightarrow J \ddot{\theta}_d(t); Bs \theta_d(s) \rightarrow B \dot{\theta}_d(t) \right]$$

The implementation of the above scheme does not require differentiation of any measured signal.

→ Control for Flexible Joints



Equation of motion:

$$J_L \ddot{\theta}_L + B_L \dot{\theta}_L + K(\theta_L - \theta_m) = 0$$

$$J_m \ddot{\theta}_m + B_m \dot{\theta}_m + K(\theta_L - \theta_m) = u$$

u : motor torque.

(Simplified model)

Here the input u is considered as motor torque instead of motor armature input voltage.

• Writing these equations in state space form:

$$\text{Let } x_1 = \theta_l \quad x_3 = \theta_m$$

$$x_2 = \dot{\theta}_l \quad x_4 = \dot{\theta}_m$$

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{\tilde{K}}{J_l} x_1 - \frac{B_l}{J_l} x_2 + \frac{\tilde{K}}{J_l} x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{\tilde{K}}{J_m} x_1 - \frac{B_l}{J_m} x_4 - \frac{\tilde{K}}{J_m} x_3 + \frac{1}{J_m} u$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\tilde{K}}{J_l} & -\frac{B_l}{J_l} & \frac{\tilde{K}}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\tilde{K}}{J_m} & 0 & -\frac{\tilde{K}}{J_m} & -\frac{B_m}{J_m} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix}}_b u$$

By choosing θ_l as output, we have:

$$y = \underbrace{[1 \ 0 \ 0 \ 0]}_C x$$

The t. f. $\frac{\theta_l(s)}{U(s)} = C (sI - A)^{-1} b$

→ The pair (A, b) on the system is completely controllable, i.e.

$$\text{rank} \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = 4$$



One can always design a controller vector k s.t. the following control signal:

$$u = k^T x + r_f$$

r_f : reference signal

can be implemented to achieve desired closed loop response.

With the above u , the closed loop system:

$$\dot{x} = \underbrace{(A + bk^T)}_{\text{closed loop system matrix}} x + br_f$$

One can design k appropriately s.t. the eigenvalues of $(A + bk^T)$ are at some desired locations.

→ A Formula for Design of 'k':

Let the desired closed-loop poles/eigenvalues of $(A+bk)$ be: $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

- First, construct a polynomial $\sigma(s)$ as follows using the chosen $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$:

$$\begin{aligned}\sigma(s) &= (s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n) \\ &= s^n + \sigma_1 s^{n-1} + \sigma_2 s^{n-2} + \dots + \sigma_{n-1} s + \sigma_n\end{aligned}$$

- Form a coefficient vector:

$$\sigma = [\sigma_1 \quad \sigma_2 \quad \dots \quad \sigma_{n-1} \quad \sigma_n]$$

- Compute the open loop characteristic polynomial:

$$\begin{aligned}\alpha(s) &:= \det(sI - A) \\ &= s^n + \alpha_1 s^{n-1} + \dots + \alpha_{n-1} s + \alpha_n\end{aligned}$$

Define a vector

$$\alpha := [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n]$$

- Using the coefficients of α_i , construct the following matrix:

$$M := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_1 & 1 & 0 & \dots & 0 \\ \alpha_2 & \alpha_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \dots & 1 \end{bmatrix} \leftarrow \text{It is a non-singular matrix!}$$

- The feedback gain vector is:

$$k^T = (\sigma - \alpha)(M^{-1})^T e^{-1}$$

where $e := [b \quad Ab \quad \dots \quad A^{n-1}b]$

Note that e is invertible under controllability assumption.

For our robotics case, set $n = 4$.

Then compute feedback gain k .

- With the resulting k , the eigenvalues of $(A + bk)$ are

$$\{\lambda_1, \lambda_2, \dots, \lambda_n\}, \text{ which}$$

are desired closed-loop poles.

→ A Useful Model for Advanced Control :

- Recall from the Robot dynamics that the equation of motion for a robot is :

$$\sum_{j=1}^n d_{kj}(q) \ddot{q}_j + \sum_{i=1}^n \sum_{j=1}^n c_{ijk}(q) \dot{q}_i \dot{q}_j + g_k = \tau_k \quad \dots \textcircled{D1}$$

for $k = 1, 2, \dots, n$ (n -link manipulator)

- Considering the DC motors attached with each link as actuator, the actuator dynamics :

$$J_{mk} \ddot{\theta}_{mk} + B_k \dot{\theta}_{mk} = \frac{K_{mk}}{R_k} V_k - \frac{\tau_k}{r_k} \quad \dots \textcircled{A1}$$

where $B_k = B_{mk} + \frac{K_{bk} K_{mk}}{R_k}$

r_k : gear ratio for k^{th} link

Note that the motor angle θ_{mk} & joint angle q_k are related by :

$$\theta_{mk} = r_k q_k$$

Now multiplying gear ratio r_k to (A1), we have:

$$r_k^2 J_{m_k} \ddot{q}_k + r_k^2 B_k \dot{q}_k = r_k \frac{K_{m_k}}{R_k} V_k - \tau_k$$

$$\Rightarrow \tau_k = \frac{r_k K_{m_k}}{R_k} V_k - r_k^2 J_{m_k} \ddot{q}_k + r_k^2 B_k \dot{q}_k \quad \dots (*)$$

Using (*) in (D1), we have:

$$r_k^2 J_{m_k} \ddot{q}_k + \sum_{j=1}^n d_{kj} \ddot{q}_j + \sum_{i,j=1}^n c_{ijk} \dot{q}_i \dot{q}_j + r_k^2 B_k \dot{q}_k + g_k = \frac{r_k K_{m_k}}{R_k} V_k$$

↓ compactly written as:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + B \dot{q} + g(q) = u$$

where $M(q) = D(q) + J$ with $J = \begin{bmatrix} r_1^2 J_{m_1} & & 0 \\ & \ddots & \\ 0 & & r_n^2 J_{m_n} \end{bmatrix}$

The input vector 'u' has components:

$$u_k = \frac{r_k K_{m_k}}{R_k} V_k \quad \text{for } k = 1, 2, \dots, n.$$

↑
input torque which depends on k^{th} motor
armature voltage V_k .

- The above model for robotic manipulation is used for advanced control such as:

- Multivariable Control
- Feedback linearization based Control
- Adaptive & Robust Control
- Passivity based Control
- Geometric Control .

Will be
taught in
Advanced
Robotics
Course

_____ X _____ O _____ X _____