

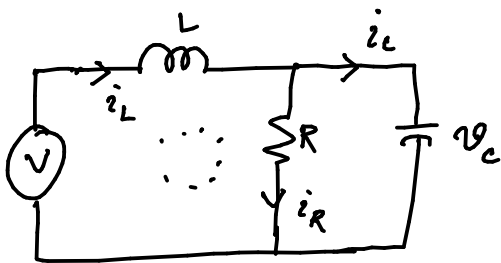
Lecture-6

Similar Systems (Equivalent systems):

$$\begin{array}{l} \text{Rep-1} \left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right. \longrightarrow \text{Define a new state variable through some non-singular transformation matrix} \\ \text{Rep-2} \left\{ \begin{array}{l} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z \end{array} \right. \quad z := V^{-1}x \end{array}$$

where $\tilde{A} = V^{-1}AV$ $\tilde{B} = V^{-1}B$ $\tilde{C} = CV$

* Rep-1 & Rep-2 of a same system have same transfer funⁿ (matrix) .



Rep-1

Assume that following quantities are state variables for the ckt:

- inductor current i_L
- capacitor voltage v_C

Apply KVL

$$L \frac{di_L}{dt} = -v_C + v \quad \text{and using KCL}$$

$$i_L = i_R + i_C$$

$$\Rightarrow i_L = \frac{v_C}{R} + C \frac{dv_C}{dt}$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v$$

Assume inductor current as op :

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

Ref-1

→ Ref-2

Let us now choose following quantities as state variables :

- inductor current i_L
- resistor current i_R

$$\frac{di_L}{dt} = -\frac{R}{L} i_R + \frac{1}{L} v$$

$$i_C = i_L - i_R$$

$$v_C = i_R R$$

$$\Rightarrow \frac{1}{C} \int i_C dt = i_R R$$

$$\Rightarrow R \frac{di_R}{dt} = \frac{1}{C} (i_L - i_R)$$

$$\Rightarrow \frac{di_R}{dt} = \frac{1}{RC} (i_L - i_R)$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{di_R}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{R}{L} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ i_R \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ i_R \end{bmatrix}$$

Ref-2

$$\begin{bmatrix} i_L \\ i_R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/R \end{bmatrix}}_V \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

Let us choose $L = 1$, $C = 1$, $R = 0.5$.

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{di_R}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} i_L \\ i_R \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ i_R \end{bmatrix}$$

$$\begin{bmatrix} i_L \\ i_R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_V \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

$$\begin{bmatrix} 0 & -0.5 \\ 2 & -2 \end{bmatrix} = V \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} V^{-1} \dots \text{choose } V \text{ appropriately}$$

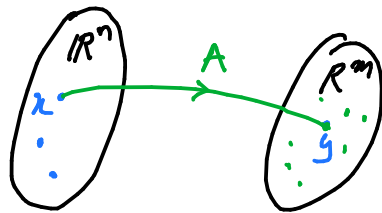
* A system can be represented in state-space form in many ways by choosing different state variables. However, all these state space representations have one transfer function.

A matrix $A \in \mathbb{R}^{m \times n}$ is a representation of a linear operator $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (or simply we can say A is a linear operator):

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

- Range Space or Image Space of A :

$$\mathcal{R}(A) := \left\{ y \in \mathbb{R}^m \mid y = Ax, x \in \mathbb{R}^n \right\}$$



$$\mathcal{R}(A) \subseteq \mathbb{R}^m$$

Column space of A
(linear combinatⁿ of
columns of A)

- Range space of A^T :

$$\mathcal{R}(A^T) := \left\{ z \in \mathbb{R}^n \mid z = A^T y, y \in \mathbb{R}^m \right\}$$

$$\mathcal{R}(A^T) \subseteq \mathbb{R}^n$$

↑
linear combinatⁿ of
rows of A

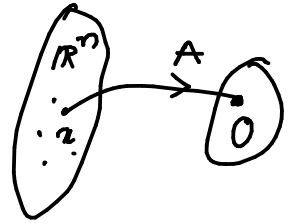
Row space

- Null space on Kernel Space of A :

$$\mathcal{N}(A) := \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$$\mathcal{N}(A) \subseteq \mathbb{R}^n$$

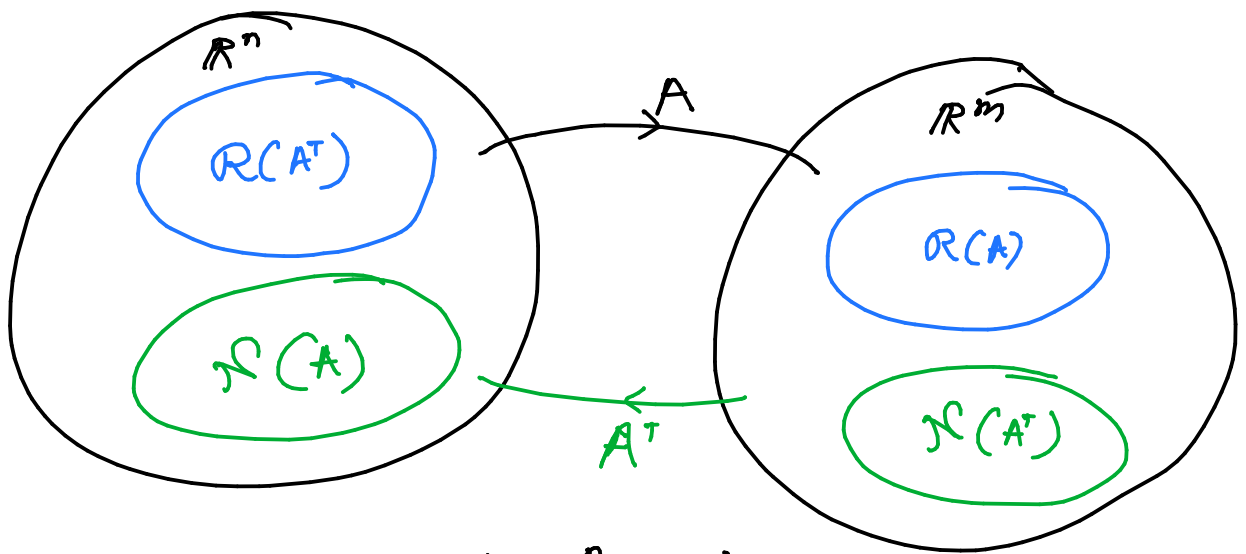
(Right null space of A)



- Null space of A^T

$$\mathcal{N}(A^T) := \{ y \in \mathbb{R}^m \mid A^T y = 0 \}$$

$$\subseteq \mathbb{R}^m$$



$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$A^T: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

- Characteristic polynomial of a matrix $A \in \mathbb{R}^{n \times n}$:

$$\alpha(s) = \det(sI - A)$$

- The roots of $\alpha(s)$ are called "eigenvalues" of matrix A .

- A vector $v \in \mathbb{R}^n$ satisfying $Av = \lambda v, \dots$ (1)

where λ is an eigenvalue of A ,
is called "eigenvector" of A .

$$(A - \lambda I)v = 0$$

A vector v , which belongs to the null space
of $(A - \lambda I)$, $(\mathcal{N}(A - \lambda I))$ is an eigenvector of A .

$(A - \lambda_i I)v_i = 0$ when v_i is an eigenvector corresponding to e.v. λ_i .

* The span of the linearly independent eigenvectors corresponding to an eigenvalue λ_i is called
"Eigenspace" of λ_i .
 //

The null space of $(A - \lambda_i I)$ is eigenspace of λ_i .

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \rightarrow \underline{\lambda_1 = 1} \quad \lambda_2 = 3$$

$$A - I = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \end{bmatrix} \leftarrow \text{is an eigenvector for } \lambda_1.$$

the dimension of eigenspace of λ_i :

↓
 the number of linearly independent
 vectors in $\mathcal{N}(A - \lambda_i I)$

$$A - 3I = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of λ_2

$$\dim(\mathcal{N}(A - \lambda_1 I)) = 1$$

$$\dim(\mathcal{N}(A - \lambda_2 I)) = 1$$

$$/ \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \text{Eigenvalues : } \lambda_1 = 2 \quad \lambda_2 = 2, \lambda = 2$$

$$(A - 2I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}}_{\text{eigenvectors corresponds to eigenvalue 2.}}$$

$$\dim(\mathcal{N}(A - 2I)) = 2$$

→ The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ has k -distinct eigenvalues.

$$\alpha(s) = (s - \lambda_1)^{n_1} (s - \lambda_2)^{n_2} \dots (s - \lambda_k)^{n_k}$$

where n_i is a positive integer & $\sum_{i=1}^k n_i = n$

- The positive integer n_i is called "algebraic multiplicity" of λ_i .
A.M.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \alpha(s) = (s - 1)^2 (s - 2)$$

the A.M of 1 = 2

A.M of 2 = 1

- * The dimension of the eigenspace of λ_i , that is $\dim(\mathcal{N}(A - \lambda_i I))$ is called "geometric multiplicity" of λ_i .
GM

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \lambda_i = 2$$

$$\text{The A.M. of } 2 \rightarrow 3$$

$$\text{G.M. of } 2 \rightarrow 2$$

$$\leftarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\rightarrow$$

$$\lambda_1 = 1 \rightarrow \text{A.M.} \rightarrow 1$$

$$\text{G.M.} \rightarrow 1$$

$$\lambda_2 = 2 \rightarrow \text{A.M.} \rightarrow 1$$

$$\text{G.M.} \rightarrow 1$$

* Result

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if it has n linearly independent eigenvectors.

* Result

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it has n -real distinct eigenvalues.

If a matrix has repeated eigenvalues ~~it~~ it is not diagonalizable

* Result

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if for every eigenvalue λ_i of A its $\text{A.M.} = \text{G.M.}$.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & -2 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\rightarrow \chi(s) = (s-1)^2(s-2)$$

$$1 \rightarrow A.M. \rightarrow 2$$

$$2 \rightarrow A.M. \rightarrow 1$$

The matrix has repeated e-vs.

$$\rightarrow \text{Corresponds to } 1 \rightarrow \text{eigenvalues} \rightarrow \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \rightarrow G.M. = 2$$

$$2 \rightarrow \text{eigenvalues} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow G.M. = 1$$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \rightarrow V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \chi(s) = (s-1)(s-2)^2$$

Repeated eigenvalues

$$1 \rightarrow A.M. \rightarrow 1$$

$$2 \rightarrow A.M. \rightarrow 2$$

$$\text{Corresponds to } 1 \rightarrow \text{eigenvalue} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow G.M. = 1$$

$$\text{Corresponds to } 2 \rightarrow \text{eigenvalue} \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \Rightarrow G.M. = 1$$

$\Rightarrow$ This matrix can not be diagonalized.

* For 1st case we got 3 linearly indep't eigenvectors $\Rightarrow$ it is diagonalizable.

→ In the second example

we have only 2 linearly eigenvectors

↓

it is not diagonalizable

↓

If a matrix is not diagonalizable, then

is it possible to obtain a matrix which
is very close to a diagonal matrix

"

block diagonal matrix ?