

Lecture - 7

* $A \in \mathbb{R}^{n \times n}$ matrix is diagonalizable if and only if it has 'n' linearly independent eigenvectors.

* Example - 1

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Ev of } A : 2, 2 \text{ \& } 1$$

$$\alpha(s) = (s-2)^2(s-1)$$

The distinct eigenvalues : 1 & 2

$$\begin{array}{lll} \text{A.M. of } 1 & \rightarrow & 1 \\ \text{" } 2 & \rightarrow & 2 \end{array} \quad \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 1 \end{array}$$

The eigenvector corresponding to $\lambda_2 = 1 \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
 v_2

The eigenvector corresponding to $\lambda_1 = 2$

$$A - 2I = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{e.v.}} \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

v_1

Is it possible to find another linearly independent e.v. corresponding to 2

* Since we do not have 3 linearly independent eigenvectors, we can not diagonalize the matrix A .

* Compute u_{11} as follows:

$$(A - 2I)u_{11} = v_1 \quad \dots \quad (*)$$

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{111} \\ u_{112} \\ u_{113} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_{113} = 0$$

$$u_{112} - 2u_{113} = 1$$

$$\Rightarrow u_{112} = 1$$

$$u_{111} = *$$

$$u_{11} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

↑ is linearly independent to v_1 & v_2

$$\text{Cust. } V = \begin{bmatrix} | & | & | \\ v_1 & u_{11} & v_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

From (*)

$$(A - 2I)u_{11} = v_1 \Rightarrow Au_{11} = 2u_{11} + v_1 \dots$$

Let us choose $\mathcal{B}_n = \{v_1, u_{11}, v_2\}$ as an ordered basis for $\mathbb{R}^3$.

New representation of matrix A w.r.t. new basis $\mathcal{B}_n$.

$$Av_1 = (\lambda_1)v_1 + (\underline{0})u_{11} + (\underline{0})v_2 \quad Av_1 = \lambda_1 v_1$$

$$Au_{11} = (\underline{1})v_1 + (\lambda_1)u_{11} + (\underline{0})v_2 \quad Au_{11} = \lambda_1 u_{11} + v_1$$

$$Av_2 = (\underline{0})v_1 + (\underline{0})u_{11} + (\lambda_2)v_2 \quad Av_2 = \lambda_2 v_2$$

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \longrightarrow \tilde{A} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \quad \begin{aligned} J_1 &= \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \\ J_2 &= \lambda_2 \end{aligned}$$

$J_i \rightarrow$ Jordan blocks of A

Jordan block

$$J = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ 0 & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

* Every square matrix $A \in \mathbb{R}^{n \times n}$ can be

represented equivalently as

$$A = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix}$$

where $J_i \rightarrow$ Jordan blocks

Jordan block of size 3

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda I + N \quad \uparrow \text{ nilpotent matrix}$$

$N^{k-1} \neq 0$ & $N^k = 0$ for some positive k

then N is called a Nilpotent matrix &
 k is index of Nilpotent matrix.

$$(A - \lambda I) v_1 = 0$$

$$(A - \lambda I) u_1 = v_1$$

$$\Rightarrow (A - \lambda I)^2 u_1 = (A - \lambda I) v_1 = 0$$

$v_1 \Rightarrow$ Eigenvector of A corresponding to e.v. λ ,

$u_1 \rightarrow$ Generalized Eigenvector of A corresponding to e.v. λ ,

* A vector u which satisfies

$$(A - \lambda I)^k u = 0 \quad \text{for } k = 2, 3, \dots$$

then u is called a generalized eigenvector of A corresponding to e.v. λ .

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ v_1 & v_1 & v_2 \end{bmatrix}$$

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example - 2

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Evs: 2, 2, 2, 2, 2

A.M. of 2 $\rightarrow$ 5

G.M. of 2 $\rightarrow$ 2

(the dimension of
null spn of
 $A - 2I$)

$$A - 2I = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Two linearly independent eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

* Since there are only 2 linearly independent eigenvectors, we can not diagonalize, so we look for Jordan canonical form.

$$(A - 2I)u_1 = 0$$

$$\textcircled{1} \quad \dots (A - 2I)u_{11} = v_1 \rightarrow u_{11} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textcircled{2} \quad \dots (A - 2I)u_{12} = u_{11} \rightarrow u_{12} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\times (A - 2I)u_{13} = u_{12}$$

Not possible to get u_{13}

$$\text{From } \textcircled{1} \quad (A - 2I)^2 u_{11} = (A - 2I)v_1 = 0 \quad \checkmark$$

$$\textcircled{2} \quad (A - 2I)u_{12} = u_{11}$$

$$(A - 2I)^2 u_{12} = (A - 2I)u_{11}$$

$$(A - 2I)^3 u_{12} = (A - 2I)^2 u_{11} = 0$$

$(A - 2I)v_1 = 0$	$\leftrightarrow$	$Av_1 = \lambda v_1$
$(A - 2I)^2 u_{11} = 0$	$\leftrightarrow$	$Au_{11} = \lambda u_{11} + v_1$
$(A - 2I)^3 u_{12} = 0$	$\leftrightarrow$	$Au_{12} = \lambda u_{12} + u_{11}$

$$\left\{ \begin{array}{l} (A - 2I)v_2 = 0 \rightarrow Av_2 = \lambda v_2 \\ (A - 2I)u_{2,1} = v_2 \rightarrow Au_{2,1} = \lambda u_{2,1} + v_2 \end{array} \right.$$

$$\hookrightarrow u_{2,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{array}{l} (A - 2I)v_2 = 0 \\ (A - 2I)^2 u_{1,2} = 0 \end{array}}$$

Construct $V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ v_1 & u_{1,1} & u_{1,2} & v_2 & u_{2,1} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$$\tilde{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$\mathcal{B}_n = \left\{ \underbrace{v_1, u_{1,1}, u_{1,2}}_{\text{Jordan chain corresponding to eigenvector } v_1}, \underbrace{v_2, u_{2,1}}_{\text{Jordan chain corresponding to eigenvector } v_2} \right\}$$

Jordan chain
corresponding to
eigenvector v_1

Jordan chain
corresponding to eigenvector v_2

Possible blocks $\rightarrow \tilde{A} = \begin{bmatrix} \times & & \\ 4 \times 4 & & \\ - & & 1 \times 1 \end{bmatrix}, \begin{bmatrix} 3 \times 3 & & \\ - & & 1 \times 1 \\ - & & 2 \times 2 \end{bmatrix}$

* The dimension of each eigen space on the geometric multiplicity of an eigenvalue λ_i will give us information about the Number of Jordan blocks associated with λ_i .

→ Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic polynomial

$$\det \alpha(s) = \det (sI - A)$$

$$= s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1s + \alpha_0$$

$$\alpha(A) = A^n + \alpha_{n-1}A^{n-1} + \dots + \alpha_1A + \alpha_0 = 0$$

For 1st example:

$$\alpha(s) = (s-2)^2(s-1)$$

$$\alpha(A) = (A-2I)^2(A-I) = 0$$

For 2nd example

$$\alpha(s) = (s-2)^5$$

$$\alpha(A) = (A-2I)^5 = 0$$

→ Is there any lower degree polynomial $m(s)$ s.t. $m(A) = 0$?

Lecture-8

$$A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \alpha(s) = (s-2)^2(s-1)$$

→ Cayley Hamilton Theorem.

$$\alpha(A_1) = (A_1 - 2I)^2(A_1 - I) = 0$$

$$\tilde{A}_1 = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \end{array} \right]$$

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \rightarrow \alpha(s) = (s-2)^5$$

↓ C.H.T

$$\alpha(A) = (A - 2I)^5 = 0$$

$$\tilde{A}_2 = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right]$$

Does there exist
other lower degree
polynomial $m(s)$ in
comparison
to characteristic
polynomial. s.t.
 $m(A) = 0$

$$\rightarrow (A - 2I)^2 (A - I) = 0 \quad (\text{No lower degree polynomial found})$$

$$\rightarrow \left. \begin{aligned} (A - 2I)^5 &= 0 \\ (A - 2I)^4 &= 0 \\ (A - 2I)^3 &= 0 \\ (A - 2I)^2 &\neq 0 \end{aligned} \right\}$$

$$\alpha(A) = (A - 2I)^5 = 0$$

$$m(A) = (A - 2I)^3 = 0$$

↑
Minimal polynomial

* Minimal polynomial :

The lowest degree polynomial $m(s)$ is called minimal polynomial of A if

$$m(A) = 0.$$

Characteristic polynomial

$$\alpha(s) = s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \dots + \alpha_1 s + \alpha_0$$

↓
C.H.T

$$A^n + \alpha_{n-1}A^{n-1} + \alpha_{n-2}A^{n-2} + \dots + \alpha_1 A + \alpha_0 = 0$$

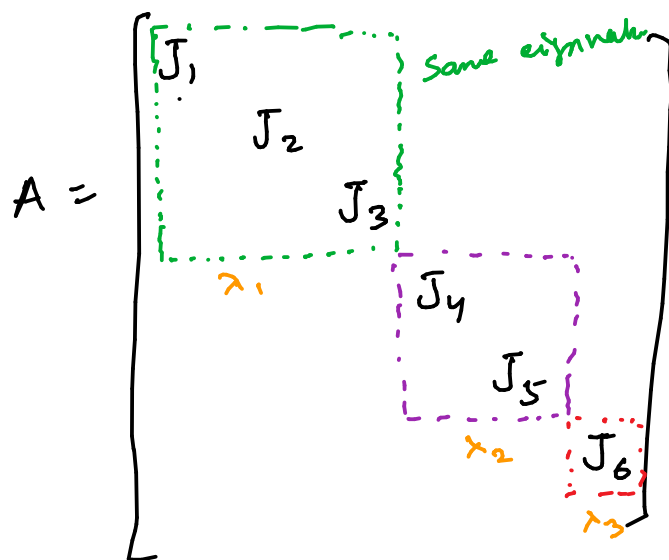
↓

Minimal polynomial

$$m(s) = s^k + \beta_{k-1}s^{k-1} + \dots + \beta_1 s + \beta_0$$

$$A^k + \beta_{k-1}A^{k-1} + \dots + \beta_1 A + \beta_0 = 0 \quad \checkmark$$

for $k < n$, $m(s)$ is the lowest degree polynomial.



$$\{J_1, J_2, J_3\} \rightarrow \text{same eigenvalue } \lambda_1$$

$$\{J_4, J_5\} \rightarrow \text{same eigenvalue } \lambda_2$$

$$\{J_6\} \rightarrow \text{same eigenvalue } \lambda_3$$



3 - distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$

- Corresponding to $\lambda_1 \rightarrow$ 3 - Jordan blocks
with J_1 is of largest size μ_1 ,
 J_2 & J_3 have size $< \mu_1$

- Corresponding to $\lambda_2 \rightarrow$ 2 - Jordan blocks
with J_4 is of largest size μ_2

- Corresponding to $\lambda_3 \rightarrow$ 1 - Jordan block with size μ_3

Then the minimal polynomial

$$m(s) = (s - \lambda_1)^{\mu_1} (s - \lambda_2)^{\mu_2} (s - \lambda_3)^{\mu_3}$$

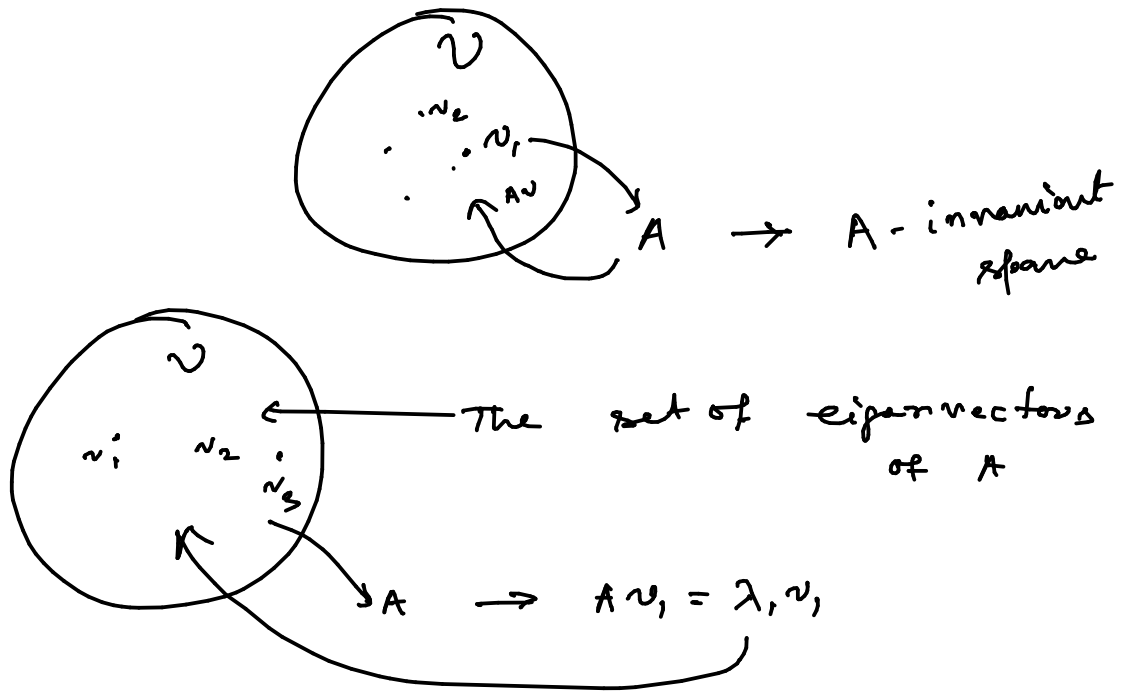
$$m(A) = (A - \lambda_1 I)^{\mu_1} (A - \lambda_2 I)^{\mu_2} (A - \lambda_3 I)^{\mu_3} = 0$$

The characteristic polynomial

$$\alpha(s) = (s - \lambda_1)^{n_1} (s - \lambda_2)^{n_2} (s - \lambda_3)^{n_3}$$

where $\mu_1 < n_1$ $\mu_2 < n_2$ $\mu_3 = n_3$

* Invariant Subspace :



————— χ —————