

L-5
Continuing

For a given matrix A

$$Av_i = \lambda_i v_i \quad \dots \quad (*)$$

Any vector v_i & scalar λ_i , satisfying $(*)$
are called :

v_i eigenvector corresponding to eigenvalue λ_i

→ roots of $\alpha(s) := \det(sI - A)$

$$Av_i = \lambda_i v_i$$

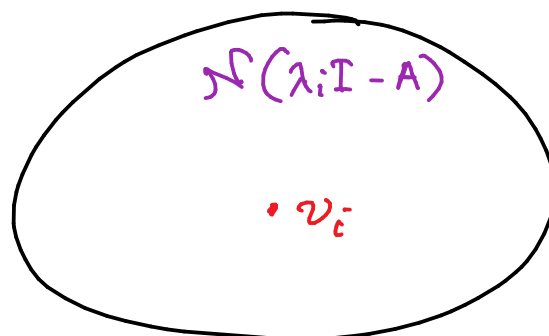
↓

$$(\lambda_i I - A)v_i = 0 \quad \dots \quad (**)$$

Any vector x satisfying

$$Ax = 0$$

The eigenvector $v_i \in \mathcal{N}(\lambda_i I - A)$ $x \in \mathcal{N}(A)$



← How to generate this subspace?

↑ "Eigen space"
"

span of linearly independent eigenvectors corresponding to λ_i .

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{eigenvalues} \quad \lambda_1 = 1 \\ \lambda_2 = 2$$

$$\lambda_1 I - A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right\} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$(\lambda_i I - A) v_i = 0$$

→ Eigenspace : The span of linearly independent eigenvectors v_i corresponding to eigenvalue λ_i

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \rightarrow \text{eigenvalues : } \lambda_1 = 1 \\ \lambda_2 = 2$$

The eigenvector corresponding to $\lambda_1 = 1$

$$I - \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & -1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

$\lambda_1 I - A$

Eigen space
 E_{λ_1}
 $\text{span} \left\{ \begin{bmatrix} * \\ 0 \end{bmatrix} \right\}$

$$\lambda_2 = 2 \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \rightarrow v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = 1$$

$$\lambda_1 I - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

- Repeated eigenvalues, but two linearly independent eigenvectors.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

$$2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}}$$

Only two linearly independent eigenvectors corresponding to eigenvalue 2.

characteristic Polynomial of A
 $\alpha(s) = \det(sI - A)$

→ Algebraic Multiplicity of an Eigenvalue:

$$\alpha(s) = (s - \lambda_1)^{\mu_1} (s - \lambda_2)^{\mu_2} \cdots (s - \lambda_k)^{\mu_k}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \rightarrow \alpha(s) = (s - 1)(s - 2)$$

$$\mu_1 = 1 \quad \mu_2 = 1$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \alpha(s) = (s - 2)^2 (s - 1)$$

$$\mu_1 = 2 \quad \mu_2 = 1$$

μ_i is algebraic multiplicity of λ_i
A.M.

A.M of 2 = 2

A.M of 1 = 1

→ Geometric Multiplicity of λ_i

($\hookrightarrow$ the number of linearly independent eigenvectors v_i corresponding to eigenvalue λ_i

$$\begin{array}{c} \text{Eigenspace} \\ E_s \\ \text{span} \{v_i^1, v_i^2, \dots\} \\ N(\lambda_i I - A) \end{array}$$

|||

dimension of eigenspace corresponding to λ_i

$$\text{i.e.} - \dim(N(\lambda_i I - A))$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ A$$

$$\rightarrow \lambda_1 = \lambda_2 = 1 \quad \lambda_3 = 2$$

$$\rightarrow \alpha(s) = (s-1)^2 (s-2)$$

A.M. of λ_1 is 2

A.M. of λ_3 is 1

$$\lambda_1 I - A = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

G.M. of $\lambda_1 = 1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \alpha(s) = (s-2)^3$$

A.M. of 2 = 3

G.M. of 2 = 2

	eigen-values	Lin. ind. eigenvectors	A.M.	G.M.
$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$	$\lambda_1 = 1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$		1	1
	$\lambda_2 = 2 \rightarrow \begin{bmatrix} -3 \\ 1 \end{bmatrix}$		1	1
$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = 1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$		2	2
	$\lambda_2 = 1$	—	—	—
$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\lambda_1 = 2 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$		3	2 ← Not equal
	$\lambda_2 = 2$			
	$\lambda_3 = 2$			

- For distinct eigenvalues, we could get same number of lin. ind. eigenvectors as the no. of eigenvalues.
- For repeated case
lin. ind. eigenvectors
 - We may get same no. of lin. ind. eigenvectors as the no. of eigenvalues
 - May not get same as the no. of eigenvalues

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \rightarrow \lambda_1 = 1 \rightarrow v_1 = \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2 \rightarrow v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

→ Find a similar matrix $\tilde{A}$ for
the choice of basis = $\{v_1, v_2\}$

$$Av_1 = (1)v_1 + (0)v_2 \quad Av_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Av_2 = (0)v_1 + (2)v_2 \quad \begin{cases} Av_1 = \\ Av_2 = \lambda_2 v_2 \end{cases}$$

$$A \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \leftarrow \text{Diagonalization}$$

$$\rightarrow \text{For any } A \in \mathbb{R}^{n \times n} \rightarrow \begin{matrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \downarrow & \downarrow & & \downarrow \\ v_1 & v_2 & \dots & v_n \end{matrix}$$

$$\boxed{Av_i = \lambda_i v_i} \text{ for all } i$$

$$Av_1 = (\lambda_1)v_1 + (0)v_2 + \dots + (0)v_n$$

$$Av_2 = (0)v_1 + (\lambda_2)v_2 + \dots + (0)v_n$$

⋮

$$Av_n = (0)v_1 + (0)v_2 + \dots + (\lambda_n)v_n$$

$$A \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}}_V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}}_{\tilde{A}}$$

$$\tilde{A} = V^{-1}AV$$

$$A \begin{bmatrix} v_2 & v_1 \end{bmatrix} = \begin{bmatrix} v_2 & v_1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_2 & \\ & \lambda_1 \end{bmatrix}}_{\tilde{A}}$$

→ The similar matrix $\tilde{A}$ depends on

- (i) the choice of basis and
- (ii) the order in which they are arranged in constructing the transformation matrix V .

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$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \rightarrow$$

$$\lambda_1 = 1 \rightarrow A.M. \rightarrow 1$$

$$\lambda_2 = 3 \rightarrow A.M. = 1$$

	A.M.	G.M.	
$\lambda_1 = 1$	1	1	$\rightarrow \text{e.v. } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lambda_2 = 3$	1	1	$\rightarrow \text{e.v. } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow$ Since there are 2 lin. independent eigenvectors $\rightarrow$ the matrix can be diagonalized.

Facts

1. A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if it has n -linearly independent eigenvectors.

2. A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if for every eigenvalue λ_i of A it is $A.M. = G.M.$

For every $\lambda_i : A.M. = G.M. \Leftrightarrow A$ has n lin. ind. eigenvectors

$\Downarrow$
One can construct V
s.t. $V^{-1}AV = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$

- A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it has ' n '-distinct real eigenvalues.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow$$

	A. M.	G. M.
$\lambda = 2$	3	2

$\Downarrow$
A has only two linearly independent eigenvectors

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & -2 \\ 0 & 3 & -1 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 1$$

2-distinct real eigenvalues

$$\alpha(s) = (s-1)^2(s-2)$$

	A. M.	G. M.
$\lambda_1 = \lambda_3 = 1$	2	2
$\lambda_2 = 2$	1	1

$$\begin{aligned} 1 &\rightarrow \text{eigenvectors} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \\ 2 &\rightarrow \quad \quad \quad = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \quad \left. \vphantom{\begin{aligned} 1 &\rightarrow \text{eigenvectors} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} } \right\} \begin{array}{l} \text{3-linearly} \\ \text{independent} \\ \text{eigenvectors} \end{array}$$

- The above matrix has no 3-distinct eigenvalues, but it can still be diagonalizable since there are 3-linearly independent eigenvectors.

$$\rightarrow A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \lambda_1 = \lambda_2 = 2 \\ \lambda_3 = 1 \end{array}$$

	A. m.	G. m.	
$\lambda_1 = 2$	2	1	$\rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\lambda_3 = 1$	1	1	$\rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

⇓

'A' can not be transformed into a diagonal matrix

$$(A - \lambda_1 I) u_1 = v_1$$

"

$$A u_1 = v_1 + \lambda_1 u_1$$

$$(\lambda_1 I - A) u_1 = v_1 \rightarrow \lambda_1 u_1 - A u_1 = v_1$$

$$(2I - A) u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{A u_1 = \lambda_1 u_1 - v_1}$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftarrow \text{Arrange the basis in this form.}$$

$v_1 \quad u_1 \quad v_2$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ v_1 & u_1 & v_2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$Av_1 = (2)v_1 + (0)u_1 + (0)v_2 \quad \swarrow \quad Av_1 = 2v_1$$

$$Au_1 = (1)v_1 + (2)u_1 + (0)v_2 \quad \leftarrow Au_1 = v_1 + 2u_1$$

$$Av_2 = (0)v_1 + (0)u_1 + (1)v_2 \quad Av_2 = 1v_2$$

$$A \begin{bmatrix} 1 & 1 & 1 \\ v_1 & u_1 & v_2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ v_1 & u_1 & v_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{\tilde{A}}$
 $\uparrow$
 Block diagonal matrix

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$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 2 \quad \lambda_3 = 1$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

← only 2 linearly independent eigenvectors

$$u_1 \leftarrow (A - \lambda_1 I) u_1 = v_1 \quad \dots (*)$$

$$\Rightarrow Au_1 = v_1 + \lambda_1 u_1$$

$$V = \begin{bmatrix} | & | & | \\ v_1 & u_1 & v_2 \\ | & | & | \end{bmatrix}$$

$$A \begin{bmatrix} v_1 & u_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & u_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

v_1 & v_2 are eigenvectors

u_1 - ?

$$\text{Jordan blocks} \begin{cases} J_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \\ J_2 = \lambda_3 \end{cases}$$

$\tilde{A}$
↓
Jordan Matrix

$$\tilde{A} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix} \leftarrow \text{Block diagonal matrix}$$

$$(A - \lambda_1 I) u_1 = v_1$$

$$\Rightarrow (A - \lambda_1 I)(A - \lambda_1 I) u_1 = \underbrace{(A - \lambda_1 I) v_1}_{=0}$$

$$(A - \lambda_1 I)^2 u_1 = 0$$

↑ "generalized eigenvector"

* Any vector u_i which satisfies

$$(A - \lambda_i I)^k u_i = 0 \text{ for } k = 2, 3, 4 \dots$$

is called "generalized eigenvector" of A corresponding to eigenvalue λ_i .

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 2$$

$$\alpha(s) = (s - 2)^5$$

$$(A - 2I) v_i = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{the } \dim(N(A - 2I)) = 2$$

$\Downarrow$

Geometric multiplicity = 2

$$\{v_1, v_2\}$$

$$(A - 2I)u_{11} = v_1 \rightarrow u_{11} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A - 2I)u_{12} = u_{11} \rightarrow u_{12} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A - 2I)u_{13} = u_{12} \rightarrow u_{13} \text{ is not linearly independent to the others}$$

We could not get 5 linearly independent vectors!

So start with v_2

$$(A - 2I)u_{21} = v_2 \rightarrow u_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left. \begin{aligned} (A - 2I)^1 v_1 &= 0 \rightarrow Av_1 = 2v_1 \\ (A - 2I)u_{11} &= v_1 \rightarrow Au_{11} = v_1 + 2u_{11} \\ &\Rightarrow (A - 2I)^2 u_{11} = 0 \\ (A - 2I)u_{12} &= u_{11} \rightarrow Au_{12} = u_{11} + 2u_{12} \\ &\Rightarrow (A - 2I)^3 u_{12} = 0 \end{aligned} \right\} \begin{array}{l} \leftarrow \text{e.v.} \\ v_1 \\ \downarrow \\ u_{11} \\ \downarrow \\ u_{12} \end{array} \left. \vphantom{\begin{aligned} (A - 2I)^1 v_1 &= 0 \\ (A - 2I)u_{11} &= v_1 \\ &\Rightarrow (A - 2I)^2 u_{11} = 0 \\ (A - 2I)u_{12} &= u_{11} \\ &\Rightarrow (A - 2I)^3 u_{12} = 0 \end{aligned}} \right\} \text{g.e.v.}$$

$$\left. \begin{aligned} (A - 2I)v_2 &= 0 \rightarrow Av_2 = 2v_2 \\ (A - 2I)u_{21} &= v_2 \rightarrow Au_{21} = v_2 + 2u_{21} \\ &\Rightarrow (A - 2I)^2 u_{21} = 0 \end{aligned} \right\} \begin{array}{l} v_2 \leftarrow \text{e.v.} \\ \downarrow \\ u_{21} \leftarrow \text{g.e.v.} \end{array}$$

$$A \left[\underbrace{\begin{bmatrix} v_1 & u_{11} & u_{12} & v_2 & u_{21} \end{bmatrix}}_{\substack{\text{Chain of} \\ \text{eigen-vectors}}} \right] = \left[\begin{bmatrix} v_1 & u_{11} & u_{12} & v_2 & u_{21} \end{bmatrix} \right] \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

V

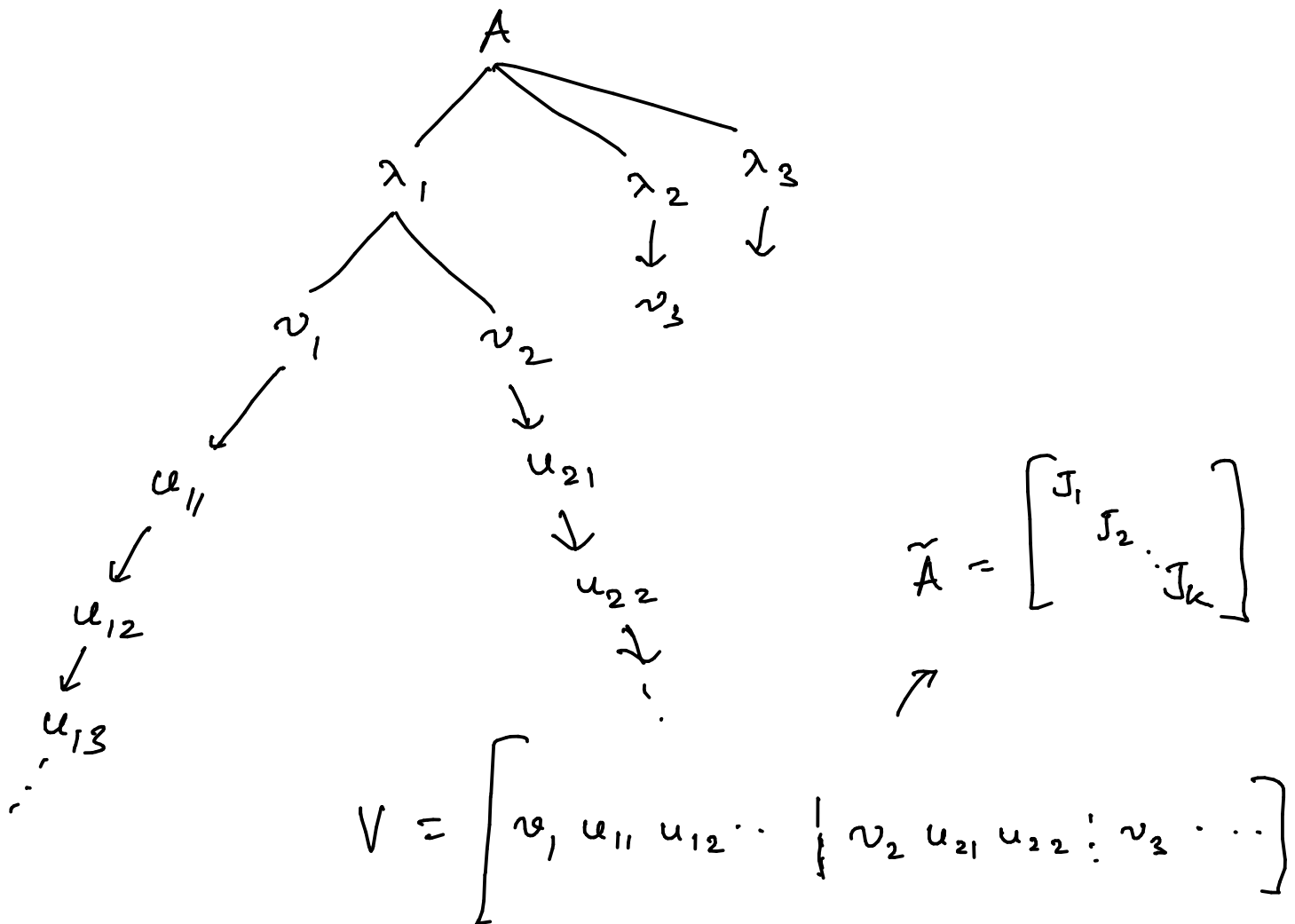
$J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$

$$A = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix}$$

$$\begin{array}{ccc} \lambda & 2 & \bar{\lambda} \\ \downarrow & & \downarrow \\ v_1, v_2 & , & v_3 \\ J_1 & J_2 & J_3 \\ \downarrow & \downarrow & \downarrow \\ 3 \times 3 & 2 \times 2 & 2 \times 2 \end{array}$$

- The sum of sizes of jordan blocks J_i corresponding to an eigenvalue λ_i is equal to the algebraic multiplicity of λ_i .

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$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

→ Characteristic Polynomial

$$\alpha(s) = \det(sI - A) \quad A \in \mathbb{R}^{n \times n}$$

$$= s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0$$

$$\alpha(A) = A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_1 A + \alpha_0 I$$

$$= 0$$

↑
Caley-Hamilton

Theorem

$$\Rightarrow A^n = -\alpha_{n-1} A^{n-1} - \alpha_{n-2} A^{n-2} - \dots - \alpha_1 A - \alpha_0 I$$

$$\text{Basis } \{b, Ab, A^2b, A^3b\}$$

$$\dot{x} = Ax + bu$$

$$A \in \mathbb{R}^{4 \times 4}$$

$$A \rightarrow \tilde{A}$$

$$A \rightarrow b \quad Ab = (0)b + (1)Ab + (0)A^2b + (0)A^3b$$

$$A \rightarrow Ab \quad A^2b = (0)b + (0)Ab + (1)A^2b + (0)A^3b$$

$$A^3b = (0)b + (0)Ab + (0)A^2b + (1)A^3b$$

$$A^4b = (-\alpha_0)b + (-\alpha_1)Ab + (-\alpha_2)A^2b + (-\alpha_3)A^3b$$

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix}$$

$$\alpha(A) = 0$$

$$\alpha(s) = \det(sI - A)$$

↓

$$\deg \alpha(s) = n$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \alpha(s) = \det(sI - A)$$

$$= (s - 2)^3$$

$$= s^3 - 6s^2 + 12s - 8$$

$$\alpha(A) = A^3 - 6A^2 + 12A - 8I$$

Represented as

$$\alpha(A) = (A - 2I)^3$$

with A as a variable.

- Define a polynomial:

$$m(s) = (s - 2)^2 = s^2 - 4s + 4$$

Now evaluate $A^2 - 4A + 4$

$$\text{for } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow A^2 =$$

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

↑

$$4A - 4I = \begin{bmatrix} 8-4 & 4 & 0 \\ 0 & 8-4 & 0 \\ 0 & 0 & 8-4 \end{bmatrix}$$

Represent the corresponding polynomial with variable A as:

$$m(A) := (A - 2I)^2$$

$$A \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

→ characteristic polynomial

$$\alpha(s) = (s-2)^3$$

⇓

According to C.H.T

$$\downarrow (A-2I)^3 = 0 \quad \dots \textcircled{2}$$

$$m(s) := (s-2)^2$$

↓

$$m(A) = (A-2I)^2 \quad \dots \textcircled{1}$$

$$= 0$$

$$\deg(m(s)) = \underline{2}$$

$$\deg(\alpha(s)) = \underline{3}$$

$m(s)$ is called "minimal polynomial".

↓

A lower degree (w.r.t. $\alpha(s)$)

monic polynomial

$$s.t. \quad m(A) = 0$$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

characteristic polynomial

$$\alpha(s) = (s-2)^5$$

↓ C.H.T

$$\alpha(A) = (A-2I)^5 = 0$$

$$m(s) := (s-2)^3$$

$$m(A) = (A-2I)^{\textcircled{3}} = 0$$

Minimal Polynomial.

minimum degree
"the size of largest Jordan block corresponding to eigenvalue 2."

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

↓

$$J_i = \lambda_i I + N$$

$2I$

+ N
↓

Nilpotent
matrix

A matrix N is Nilpotent Matrix

$$\text{if } N^{\mu-1} \neq 0 \text{ \& } N^{\mu} = 0$$

μ is a positive integer.

μ : Nilpotent index of N

- Let for a given matrix A , its minimal polynomial

$$m(s) = (s - \lambda_1)^{r_1} (s - \lambda_2)^{r_2} \dots (s - \lambda_k)^{r_k} = 0$$

Then r_i is the size of largest Jordan block corresponding to eigenvalue λ_i .

$$m(A) = (A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \dots (A - \lambda_k I)^{r_k} = 0$$

