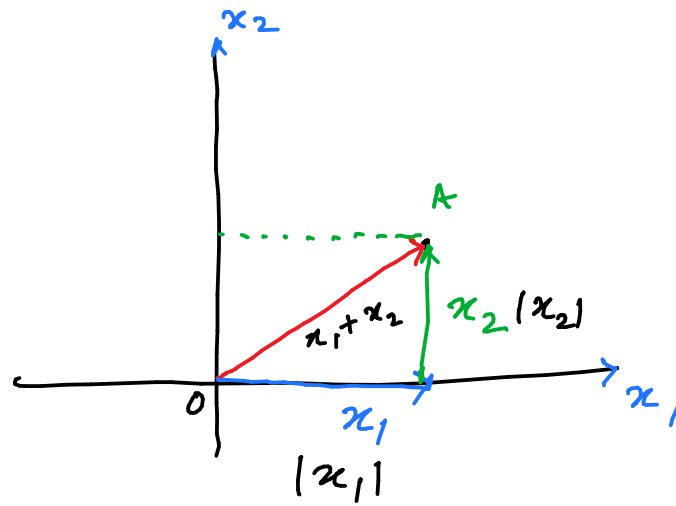


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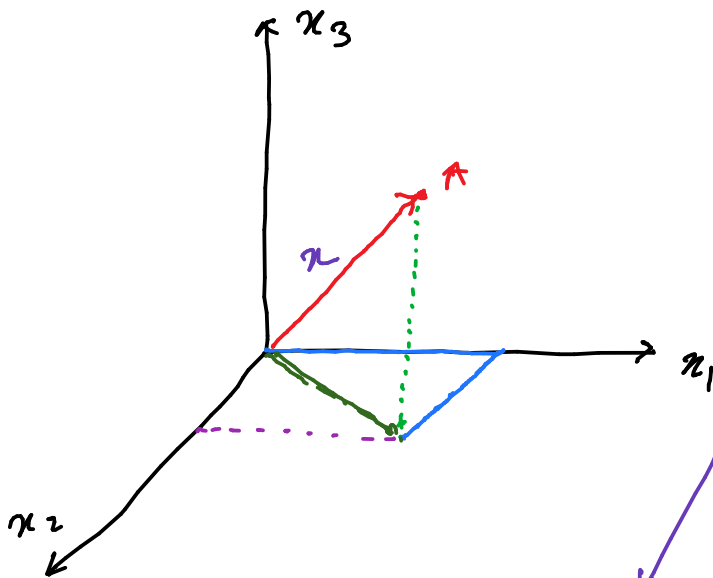


$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use Pythagorean theorem to compute what is the length or magnitude of $x_1 + x_2$

$$|OA| = |x_1 + x_2| = \sqrt{x_1^2 + x_2^2}$$

Distance from O to A



$$|OA| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\|x\| \leftarrow$ Length of x from origin

Distance to x from O.

$$\text{Let } x \in \mathbb{R}^n \rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

↙
"Norm" of x

→ The "norm" is a function:

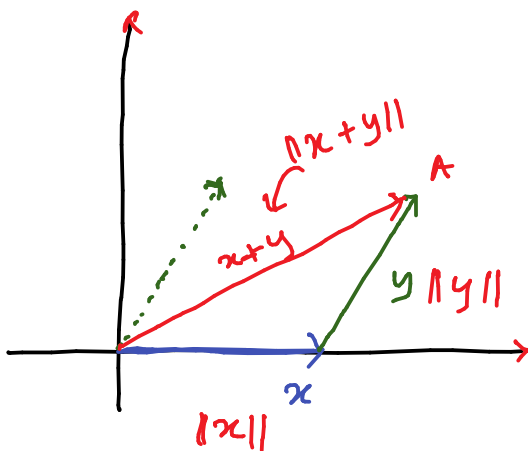
$$\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$$

- $\|\cdot\|$ is a non-negative real number

$$\|x\| \geq 0 \quad ; \quad \|x\| = 0 \text{ if } x = 0.$$

- $\|\alpha x\| = |\alpha| \|x\|$

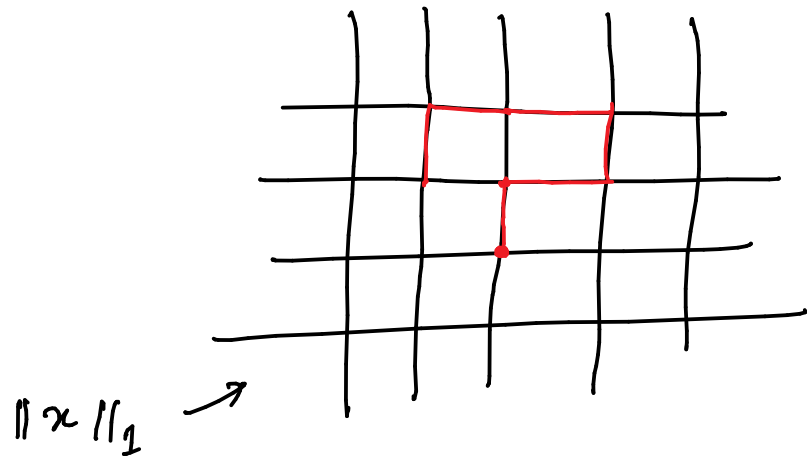
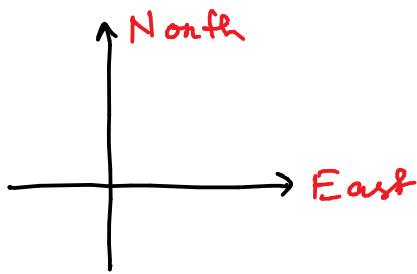
- $\|x + y\| \leq \|x\| + \|y\|$



$$\|x + y\| \leq \|x\| + \|y\|$$

Define $\|x\|_1 := \sum_{i=1}^n (x_i)$ ↙ grid-norm $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

↑
One norm



$$\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \quad \leftarrow \text{2-norm of a vector}$$

$$\|x\|_\infty := \max_i |x_i| \quad \leftarrow \text{Infinity norm}$$

p-norm of a vector: $p \geq 1$

$$\|x\|_p := \left[\sum_{i=1}^n |x_i|^p \right]^{1/p}$$

→ Cauchy-Schwarz Inequality

$$|x^T y| \leq \|x\|_2 \|y\|_2 \quad \text{for all } \begin{matrix} x \in \mathbb{R}^n \\ y \in \mathbb{R}^n \end{matrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x^T x = \|x\|_2^2$$

$$x^T y = [x_1 \ x_2 \ x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \sum_{i=1}^3 x_i y_i$$

$$\text{def } \alpha := \frac{x^T y}{\|x\|^2} \quad x \neq 0$$

$$\downarrow$$

$$x^T (\alpha x - y) = \alpha x^T x - x^T y = 0 \quad \dots (*)$$

Note that

$$\underbrace{(\alpha x - y)^T (\alpha x - y)}_{\substack{\uparrow \geq 0 \\ \| \alpha x - y \|_2^2}} = \underbrace{\alpha x^T (\alpha x - y)}_{= 0} - y^T (\alpha x - y)$$

$$= -y^T (\alpha x - y)$$

$$= y^T y - \alpha y^T x$$

$$= \|y\|_2^2 - \frac{x^T y}{\|x\|_2^2} (y^T x)$$

$$= \frac{\|y\|_2^2 \|x\|_2^2 - (x^T y)(y^T x)}{\|x\|_2^2}$$

$$= \frac{\|y\|_2^2 \|x\|_2^2 - |x^T y|^2}{\|x\|_2^2}$$

$$\Rightarrow \frac{\|y\|_2^2 \|x\|_2^2 - |x^T y|^2}{\|x\|_2^2} \geq 0$$

$$\text{since } x \neq 0 \Rightarrow \|x\|_2^2 \neq 0$$

$$\Rightarrow \|y\|_2^2 \|x\|_2^2 \geq |x^T y|^2$$

$$\Rightarrow |x^T y| \leq \|x\|_2 \|y\|_2 \quad \dots \quad \square$$

→ $\|x\|_2 := \left[\sum_{i=1}^n |x_i|^2 \right]^{1/2}$ is a norm.

• First condition holds

$$\begin{aligned} \|\alpha x\|_2^2 &= \alpha^2 (x_1^2 + x_2^2 + \dots + x_n^2) \\ &= \alpha^2 \|x\|_2^2 \end{aligned}$$

$$\Rightarrow \|\alpha x\|_2 = |\alpha| \|x\|_2$$

$$\bullet \quad \|x + y\|_2^2 = \sum_{i=1}^n (x_i + y_i)^2$$

$$= \sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2$$

$$= \sum_{i=1}^n x_i^2 + 2 x^T y + \sum_{i=1}^n y_i^2$$

$$\leq \sum_{i=1}^n x_i^2 + 2 |x^T y| + \sum_{i=1}^n y_i^2$$

$$\leq \|x\|_2^2 + 2 \|x\| \|y\| + \|y\|_2^2 \quad \downarrow \text{ by Cauchy-Schwarz inequality}$$

$$= (\|x\|_2 + \|y\|_2)^2$$

$$\Rightarrow \|x + y\|_2 \leq \|x\|_2 + \|y\|_2 \quad \checkmark$$

Hence the definition is
a valid vector norm.

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Norm of a vector $x \in \mathbb{R}^n$

$$\|x\|_2, \|x\|_1, \|x\|_\infty, \|x\|_p$$

$$\alpha \|x\|_C \leq \|x\|_A \leq \beta \|x\|_B \quad \checkmark$$

→ All norms are equivalent.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\|v_2\|_2^2 = 1^2 + 2^2 + 3^2 + 4^2$$

$$\|A\|_F^2 := \sum_{i,j} |a_{ij}|^2$$

← Frobenius Norm of $A \in \mathbb{R}^{n \times m}$

→ Matrix norm

For $A \in \mathbb{R}^{n \times m}$, matrix norm is a function $\|\cdot\| : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}_{\geq 0}$ s.t. following conditions are satisfied

$$(i) \|A\| \geq 0 \quad \& \quad \|A\| = 0 \text{ if } A = 0$$

$$(i) \quad \|\alpha A\| = |\alpha| \|A\|$$

$$(ii) \quad \|A+B\| \leq \|A\| + \|B\|$$

$$(iv) \quad \|AB\| \leq \|A\| \|B\|$$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_m \\ | & | & & | \end{bmatrix} \quad A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}$$

$a_j \leftarrow j^{\text{th}} \text{ col. of } A$ $a_i^T \leftarrow i^{\text{th}} \text{ row of } A$

$$\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_i \|A_{i*}\|_2^2 = \sum_i \|a_i^T\|_2^2 \quad \left. \begin{array}{l} * \rightarrow \text{all } j \\ A_{i*} = a_i^T \end{array} \right\}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \rightarrow \|A\|_F^2 = \|a_1^T\|_2^2 + \|a_2^T\|_2^2$$

$$\|A\|_F^2 = \sum_i \|a_i^T\|_2^2 = \sum_j \|a_j\|_2^2$$

$$\|Ax\|_2^2 = \sum_i |a_i^T x|^2 \leq \underbrace{\sum_i \|a_i\|_2^2}_{\|A\|_F^2} \underbrace{\|x\|_2^2}_{\|x\|_2^2} \leftarrow \begin{array}{l} \text{Cauchy-Schwarz} \\ \text{Inequality} \end{array}$$

$Ax = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$

$$= \|A\|_F^2 \|x\|_2^2 \leftarrow \|Ax\|_2 \leq \|A\|_F \|x\|_2$$

$$\|AB\|_F^2 \rightarrow AB = M = m_j \leftarrow j^{\text{th}} \text{ column of } M.$$

$$m_j = A b_j \quad b_j : j^{\text{th}} \text{ col. of } B.$$

$$\begin{bmatrix} m_1 & m_2 & \dots & m_k \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix}$$

B

$$\|AB\|_F^2 = \sum_j \|m_j\|_2^2 = \sum_j \|A b_j\|_2^2$$

$$\leq \|A\|_F^2 \underbrace{\sum_j \|b_j\|_2^2}_{\|B\|_F^2} \leftarrow \text{from previous result}$$

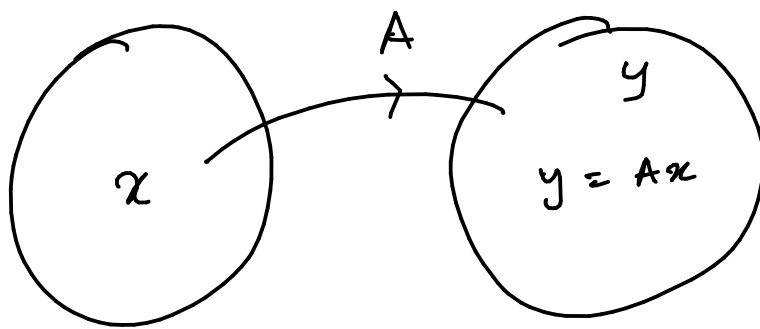
$$= \|A\|_F^2 \|B\|_F^2$$

$$\|AB\|_F \leq \|A\|_F \|B\|_F \leftarrow$$

$\rightarrow$ Let us define another matrix norm:
(Induced Matrix Norm)

$$\|A\|_m := \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

v is any vector norm
1, 2, ∞



$$\frac{\|y\|}{\|x\|} \leftarrow \text{output}$$

$$\uparrow$$

$$\leftarrow \text{input}$$

$$\downarrow$$

$$A_M := \max_x \frac{\|y\|_2}{\|x\|_2}$$

$$Av = \lambda v$$

- $\|A\|_M \geq 0$ & $\|A\|_M = 0$ if $A = 0$

if $A \neq 0$ $\|A\|_M > 0$

Since $A \neq 0$, one can find a $x \neq 0$

s.t. $Ax \neq 0$

$$\max_x \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|}$$

$\uparrow$

x which maximizes

$\uparrow$ for any $x \neq 0$

$$\frac{\|Ax\|}{\|x\|} > 0 \quad \text{for } x \neq 0$$

$$\Rightarrow \max_x \frac{\|Ax\|}{\|x\|} > 0 \Rightarrow \|A\|_M > 0_{A \neq 0}$$

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$$\|A\|_m := \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|y\|_2}{\|x\|_2}$$

where $y = Ax$

- $\|\alpha A\|_m = |\alpha| \|A\|_m$

$$\begin{aligned}\|\alpha A\|_m &= \max_x \frac{\|\alpha Ax\|_2}{\|x\|_2} = \max_x \frac{\|\alpha y\|_2}{\|x\|_2} \\&= |\alpha| \max_x \frac{\|y\|_2}{\|x\|_2} \\&= |\alpha| \|A\|_m\end{aligned}$$

- $\|A+B\|_m \leq \|A\|_m + \|B\|_m$

$$\begin{aligned}\|A+B\|_m &:= \max_x \frac{\|(A+B)x\|_2}{\|x\|_2} \\&= \max_x \frac{\|Ax + Bx\|_2}{\|x\|_2} \\&\leq \max_x \left[\frac{\|Ax\|_2}{\|x\|_2} + \frac{\|Bx\|_2}{\|x\|_2} \right]\end{aligned}$$

Handwritten green annotations:
An arrow points from x to Ax with the label $x \sim Ax$.
Another arrow points from x to Bx with the label $y = Bx$.

$$\leq \max_x \frac{\|Ax\|_v}{\|x\|_v} + \max \frac{\|Bx\|_v}{\|x\|_v}$$

$$= \|A\|_m + \|B\|_m$$

$$\Rightarrow \|A+B\|_m \leq \|A\|_m + \|B\|_m$$

$$\bullet \quad \|AB\|_m \leq \|A\|_m \|B\|_m$$

$$\frac{\|Ax\|_v}{\|x\|_v} \leq \max_x \frac{\|Ax\|}{\|x\|}$$

$$\Rightarrow \frac{\|Ax\|_v}{\|x\|_v} \leq \|A\|_m$$

$$\Rightarrow \boxed{\|Ax\|_v \leq \|A\|_m \|x\|_v}$$

$$\|(AB)\|_m = \max_x \frac{\|ABx\|_v}{\|x\|_v}$$

$$= \max_x \frac{\|Ay\|_v}{\|x\|_v}$$

$$y = Bx$$

$$\leq \max_x \left(\frac{\|A\|_m \|y\|}{\|x\|} \right)$$

$$\leq \|A\|_m \max_x \frac{\|Bx\|_v}{\|x\|_v}$$

$$= \|A\|_m \|B\|_m$$

This implies the function

$$\|A\|_m := \max_x \frac{\|Ax\|_v}{\|x\|_v} \quad \text{is a valid matrix norm.}$$

↓

Useful

$$\|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

|||

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$y \rightarrow \|y\|_2$$

$$\frac{1}{\|y\|_2} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

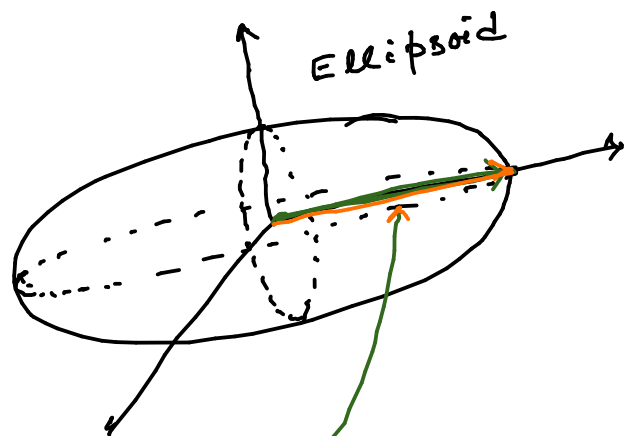
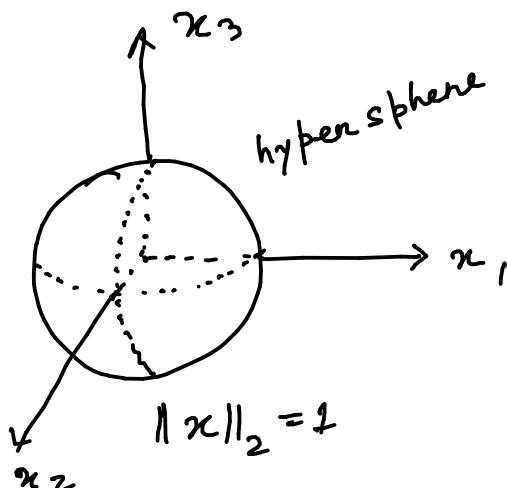
$$\|x\|_2^2 = 1$$

$$x_1^2 + x_2^2 = 1$$

$$y = Ax$$

$$\|Ax\|_2^2 = \|y\|_2^2 = y^T y$$

$$= x^T (A^T A) x \quad \checkmark$$



$\|A\|_2 \leftarrow$ length of major axis of ellipse

Ellipse equation with axes aligned with horizontal and vertical axes.

$$\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \rightarrow \quad \alpha_1^2 x^2 + \alpha_2^2 y^2 = 1$$

For $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{aligned} x^T A^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 9x_2 \end{bmatrix} \\ &= 4x_1^2 + 9x_2^2 \end{aligned}$$

$$x^T A^T A x = c \quad \rightarrow \quad 4x_1^2 + 9x_2^2 = c$$

↑ Equation of ellipse.

• $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} x^T A^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} \\ &= 2x_1^2 - 2x_1x_2 + x_2^2 \end{aligned}$$

$$x^T A^T A x = c \quad \rightarrow \quad 2x_1^2 - 2x_1x_2 + x_2^2 = c$$

↑
Tilted
ellipse

↖ Also an ellipse
Now its major & minor axes are not aligned with horizontal & vertical axes.

- $\|A\|_1 := \max_j \sum_i |a_{ij}|$

↑ Largest absolute column sum

- $\|A\|_\infty := \max_i \sum_j |a_{ij}|$

↑ Largest absolute row sum

For 2-Norm

$$\|A\|_2 = \|A^T\|_2$$

$$\|A^T A\|_2 = \|A\|_2^2$$

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_2 = \max \{ \|A\|_2, \|B\|_2 \}$$

$$\|U^T A V\|_2 = \|A\|_2$$

where $U^T U = I \quad V^T V = I$

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→ Inner Product

An inner product on real vector space V is a function that maps a pair of vectors x, y to a real number

$$\langle x, y \rangle \rightarrow \langle \cdot, \cdot \rangle : (x, y) \rightarrow \mathbb{R}$$

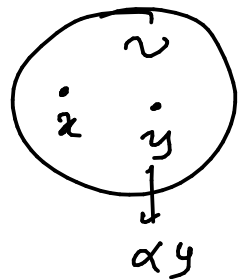
such that following properties hold

i) $\langle x, x \rangle \geq 0 \quad \& \quad \langle x, x \rangle = 0 \text{ if } x = 0$

ii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle \quad \alpha \text{ is a scalar.}$

iii) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

* iv) $\langle x, y \rangle = \langle y, x \rangle \leftarrow \text{Symmetry}$



* A vector space where an inner-product is well-defined (elements satisfies the above properties) is called

"inner-product space".

→ Dot product in Euclidean space

$$\langle x, y \rangle := \underline{x^T y} = \sum_{i=1}^n x_i y_i$$

$x \cdot y$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Euclidean
space
 x, y

$$x \cdot y = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot x$$

$$x \cdot y = \underline{x^T y} = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 4$$

- On Euclidean space with
inner-product defined as follows:

$$\langle x, y \rangle := x^T y$$

is an inner-product space.

→ vector space of all $m \times n$ matrices

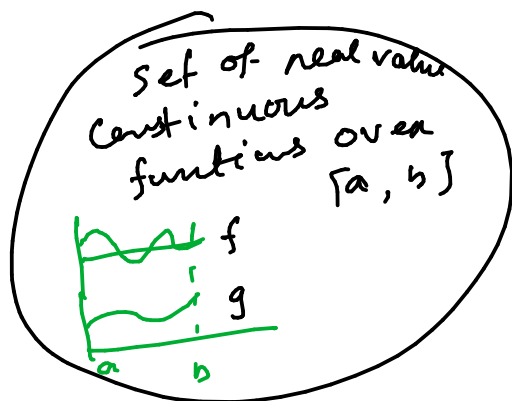
↓

Define an inner-product:

$$\langle A, B \rangle := \text{trace}(A^T B)$$

vector space
of $n \times m$
matrices
Inner product
 $\langle A, B \rangle = \text{tr}(A^T B)$

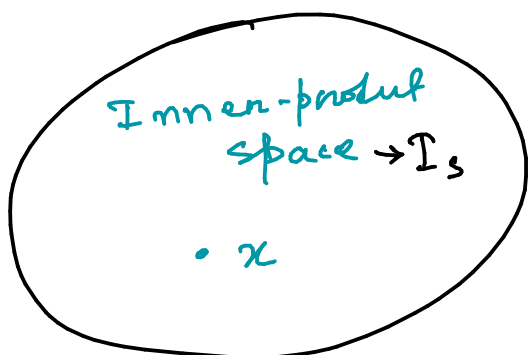
→ Vector space of real valued continuous functions over an interval $[a, b]$



Inner-product

$$\langle f, g \rangle := \int_a^b f(t)g(t) dt$$

→ Norm on an Inner-product Space



Norm of $x \in I_s$

$$\|x\| := \sqrt{\langle x, x \rangle}$$

→ Euclidean space

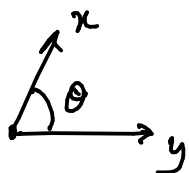
$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{x^T x}$$

→ Vector space of $m \times n$ matrices:

$$\|A\| := \sqrt{\langle A, A \rangle} = \sqrt{\text{tr}(A^T A)}$$

→ Vector space of continuous fct' over $[a, b]$

$$\|f\| := \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f^2(t) dt}$$



Dot product

$$x \cdot y = \|x\| \|y\| \cos \theta$$

$$x \cdot y = \|x\| \|y\| \cos \theta \leftarrow \text{in Euclidean space}$$



$$\langle x, y \rangle = \|x\| \|y\| \cos \theta$$

...

→ The angle between two vectors x & y in an inner-product space is a number $\theta \in [0, \pi]$

s.t.

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

← Norms over inner-product space



A particular case is on Euclidean space

$$\cos \theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

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→ Cauchy-Schwarz Inequality

$$|x^T y| \leq \|x\|_2 \|y\|_2 \quad \text{in Euclidean space}$$

in Inner product space

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

← norms are defined over inner-product sp.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

↓

$$-(\|x\| \|y\|) \leq \langle x, y \rangle \leq (\|x\| \|y\|)$$

↓

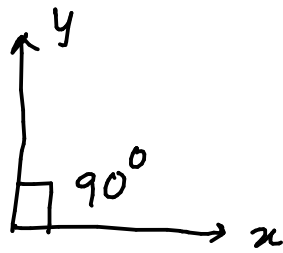
$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1$$

↑
 $\cos \theta$

$$\text{for } \theta \in [0, \pi] \rightarrow -1 \leq \cos \theta \leq 1$$

↓

→



$$\cos \theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

$$\theta = 90^\circ$$

$$\boxed{x^T y = 0}$$

$$\boxed{\langle x, y \rangle = 0}$$

⇔

x & y are orthogonal

↑
 x & y are orthogonal

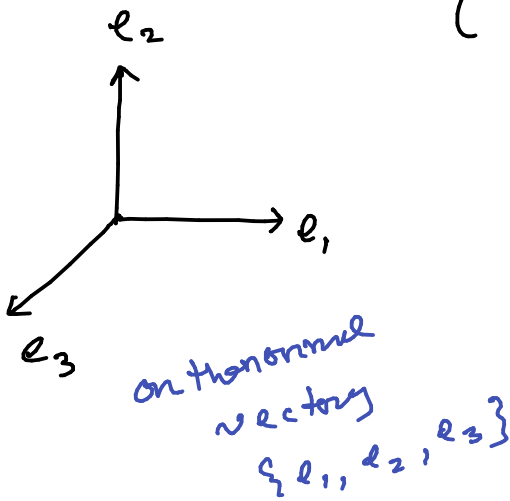
$$\left. \begin{array}{l} \|x\|_2 = 1 \\ \|y\|_2 = 1 \\ \text{or } x^T y = 0 \end{array} \right\}$$

→ x & y are

orthonormal
vectors.

Choose a set of 'n' orthonormal vectors

$$\{q_1, q_2 \dots q_n\}$$



e_1 is orthogonal to e_2 & e_3

e_3 is " " " e_1 & e_2

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ q_1 & q_2 & \dots & q_n \\ 1 & 1 & 1 \end{bmatrix} \rightarrow Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\|q_i\| = 1$$

$$q_i^T q_i = \|q_i\|_2 = 1$$

$$q_i^T q_j = 0 \quad \text{for } i \neq j$$

$$Q^T Q = I$$

← Orthogonal matrix

Orthogonal matrix Q is a matrix which satisfies $Q^T Q = I$.

$$\begin{aligned}
 & AB = I \\
 & B \text{ has inverse} = A \\
 & \rightarrow B^{-1}B = I \quad \checkmark
 \end{aligned}$$

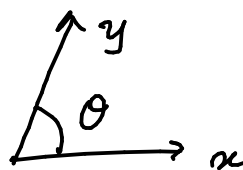
$$Q^T Q = I$$

→ The inverse of an orthogonal matrix $Q = Q^T$

→ $x \xrightarrow{Q} Qx$ Q is orthogonal matrix

$$\begin{aligned}
 \|Qx\|_2^2 &= [(Qx)^T (Qx)] \\
 &= x^T Q^T Q x \\
 &= x^T x \\
 &= \|x\|_2^2
 \end{aligned}$$

$$\begin{aligned}
 x &\rightarrow Qx \\
 y &\rightarrow Qy
 \end{aligned}$$



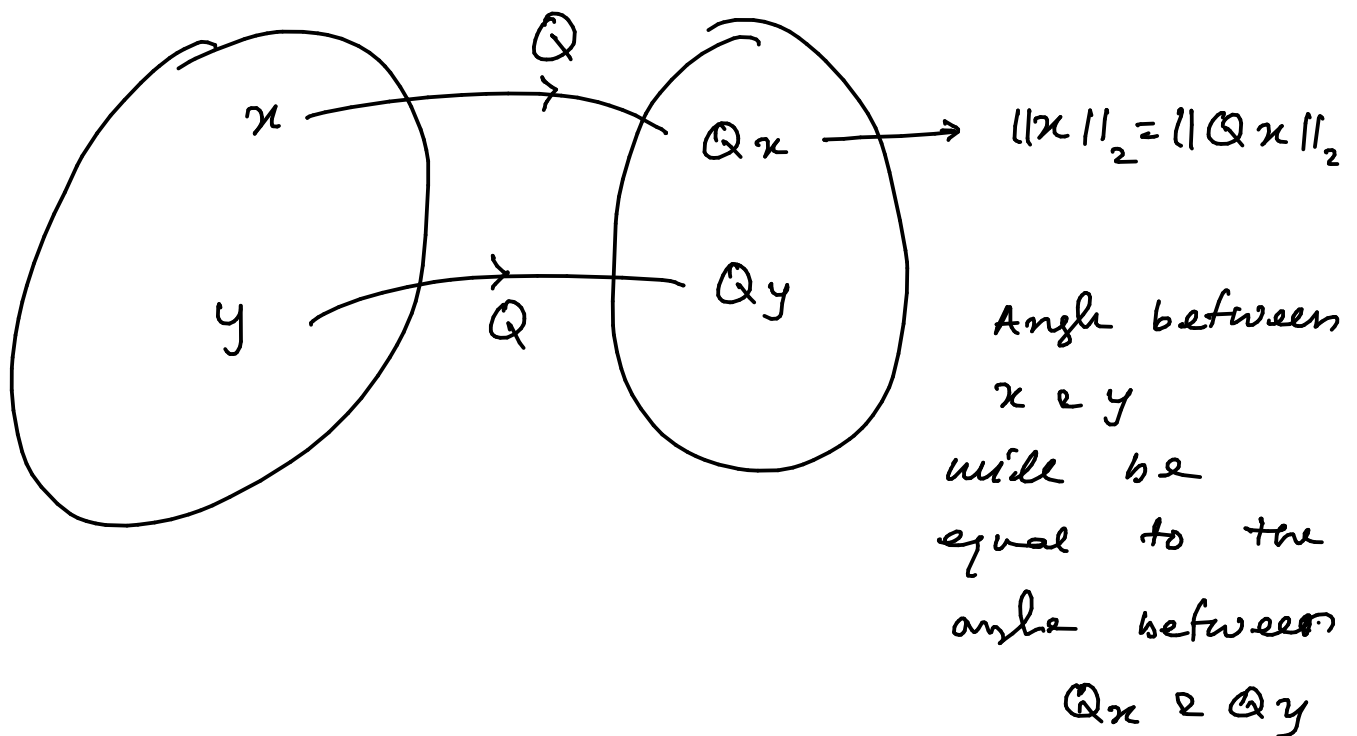
Let say the angle between x & y is θ

What is angle between Qx & Qy

$$\cos \theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

$$\frac{(Qx)^T(Qy)}{\|Qx\|_2 \|Qy\|_2} = \frac{x^T y}{\|x\|_2 \|y\|_2} = \cos \theta$$

→ The angles & the norm will get preserved under Orthogonal transform?



→ $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad ?$

$Q^T Q = ? \quad I$

↑ $\Rightarrow$ an orthogonal matrix

For $\theta = 90^\circ \rightarrow Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\theta = 45^\circ =$$

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

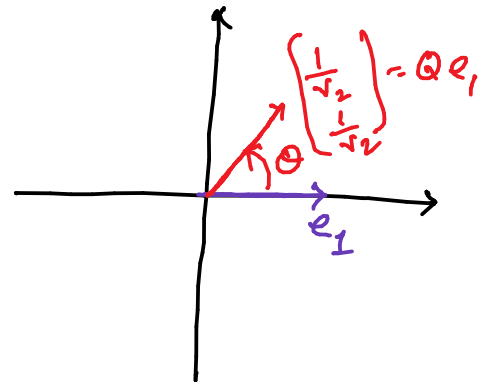
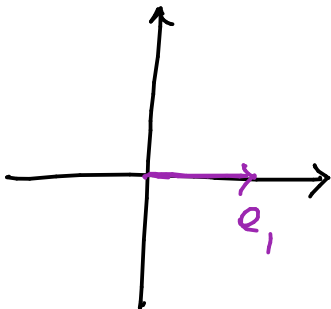
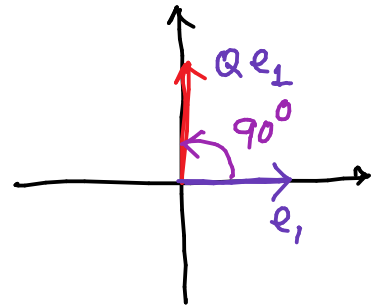
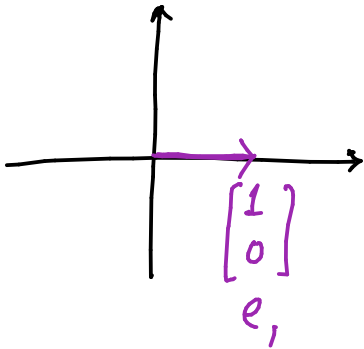
$$\theta = 90^\circ$$

$$Qe_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\theta = 45^\circ$$

$$Qe_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$



$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

↑

Rotation Matrix

NL/29/08/25

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Qx \rightarrow y$$

x will get rotated by an angle θ , & will produce y .

$$\cos \theta : c$$

$$\sin \theta : s$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

$$\text{For } x \in \mathbb{R}^3$$

$$Qx = y$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow Qe_1 = e_1$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \theta = 90^\circ$$

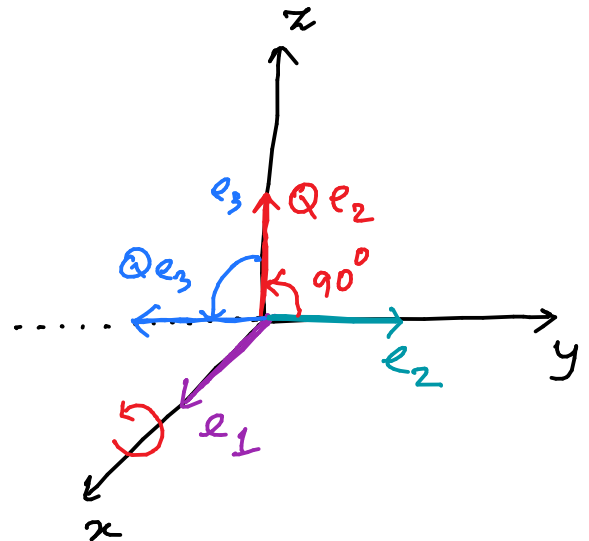
$$\rightarrow Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Qe_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow Qx$$

$$Qe_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$



$$Q = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$$\theta = 90^\circ \rightarrow Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q e_2 = e_2$$

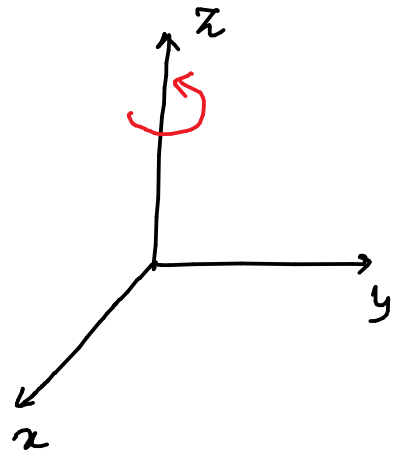
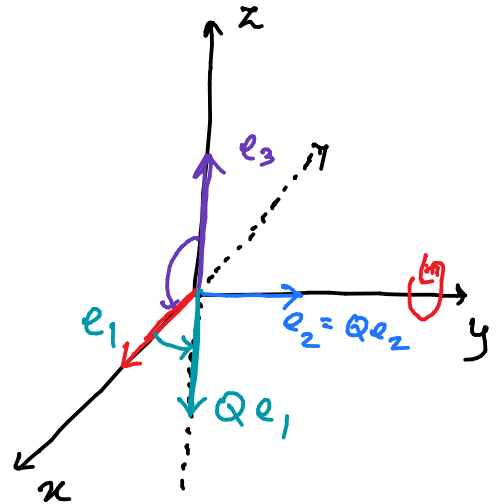
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q e_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$Q_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

← Rotation about x-axis

$$Q_y = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

← Rotation about y-axis

$$Q_z = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

← Rotation about z-axis

$$Q_{ij} = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & \dots & s \\ & & & \vdots & \ddots & \vdots \\ & & & -s & \dots & c \\ & 0 & & & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$\dots \dots \dots i^{th} \text{ position}$
 $\dots \dots \dots j^{th} \text{ position}$
 i^{th} j^{th}



Rotation Matrix ↔ Plane Rotation



Givens Rotation

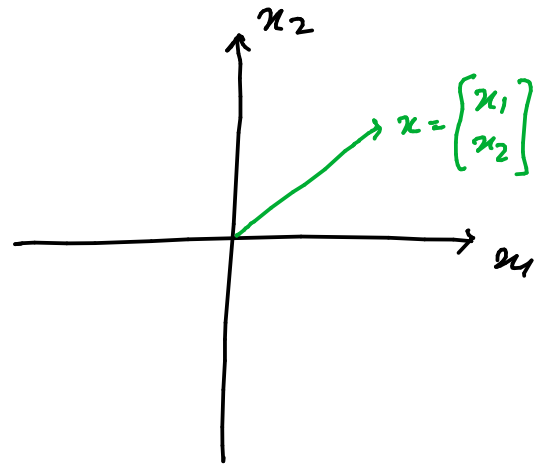


Rotation in $(i-j)^{th}$ plane; Jacobi Rotation

$$Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Qx = y = \begin{bmatrix} * \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 + sx_2 \\ -sx_1 + cx_2 \end{bmatrix}$$

← 0

$$-sx_1 + cx_2 = 0$$

$$\Rightarrow cx_2 = sx_1$$

• What would be θ in terms of x_1 & x_2

$$c = \cos \theta$$

$$s = \sin \theta$$

$$\tan \theta \rightarrow \theta?$$

$$C x_2 = S x_1$$

$$\text{Choose } \left. \begin{array}{l} C = x_1 \\ S = x_2 \end{array} \right\}$$

We need to satisfy $C^2 + S^2 = 1$

trigonometric identity

$$C = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{x_1}{\|x\|_2}$$

$$S = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{x_2}{\|x\|_2}$$

$$Q = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\|x\|} & \frac{x_2}{\|x\|} \\ -\frac{x_2}{\|x\|} & \frac{x_1}{\|x\|} \end{bmatrix}$$

$$\downarrow$$

$$y = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

$$* = C x_1 + S x_2$$

$$= \frac{x_1}{\|x\|} \cdot x_1 + \frac{x_2}{\|x\|} \cdot x_2$$

$$= \frac{x_1^2 + x_2^2}{\|x\|} = \|x\| \rightarrow y = \begin{bmatrix} \|x\| \\ 0 \end{bmatrix}$$

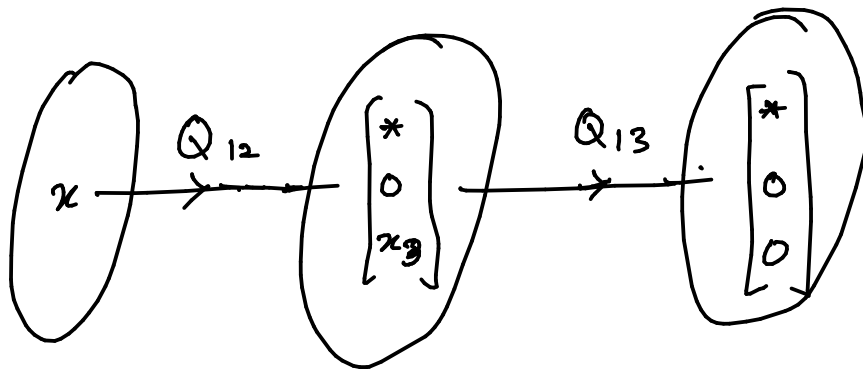
$$\text{Given } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \text{ Required}$$

$$Q \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \rightarrow Q_{12} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ x_3 \end{bmatrix}$$

$$Q_{12} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_{13} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$$Q_{13} \begin{bmatrix} * \\ 0 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$



$$Q_{12}x = \begin{bmatrix} * \\ 0 \\ x_3 \end{bmatrix}$$

$$\downarrow$$

$$\underbrace{Q_{13} Q_{12}}_Q x = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

$x \in \mathbb{R}^n$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{Q} \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$