

NL/30/08/25

Given $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Does there exist an orthogonal matrix Q s.t.

$$Qx = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

$\Delta \downarrow \cos \theta / \sin \theta$

$$\begin{bmatrix} \Delta & \Delta & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \Delta & 0 & \Delta \\ 0 & 1 & 0 \\ \Delta & 0 & \Delta \end{bmatrix} \begin{bmatrix} * \\ 0 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

$Q_{ij} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & c & \dots & s \\ & & & \vdots & 1 & \ddots \\ & & & -s & \dots & c & 1 \\ 0 & & & & & & \ddots & 1 \end{bmatrix}$

$\leftarrow$ Rotation matrix

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$Q_{ij} x = \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ c x_i + s x_j \leftarrow i^{\text{th}} \rightarrow * \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ -s x_i + c x_j \leftarrow j^{\text{th}} \rightarrow 0 \\ x_{j+1} \\ \vdots \\ x_n \end{bmatrix}$$

$$c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}} \quad s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$$



j^{th} element of $x = 0$

i^{th} element $\cdot (c x_i + s x_j)$

$$\frac{x_i^2}{\sqrt{x_i^2 + x_j^2}} + \frac{x_j^2}{\sqrt{x_i^2 + x_j^2}} = \sqrt{x_i^2 + x_j^2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{Q} \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\rightarrow$ We need
 $n-1$ Rotations

→ Every non-zero vector $x \in \mathbb{R}^n$ can be rotated to its i th coordinate $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th axis}$ by a sequence of $(n-1)$ plane rotations.

$$\rightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$\downarrow$$

$$Q_{12} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underbrace{Q_{12} A}_A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ a_{31} & * & * \end{bmatrix}$$

$$Q_{13} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \quad \underbrace{Q_{13} A}_A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

$$Q_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \quad \underbrace{Q_{23} A}_R = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$\underbrace{Q_{23} Q_{13} Q_{12}}_{Q^T} A = R \quad \downarrow$$

$$\underbrace{\underbrace{Q_{23} Q_{13}}_{A_2} Q_{12} A}_{A_1} = R$$

$$\underbrace{Q_{23} \ Q_{13} \ Q_{12}}_{Q^T} A = R$$

$$Q^T A = R$$

$$\Rightarrow A = QR \quad \leftarrow \text{QR - Decomposition of } A$$

Where Q is an orthogonal matrix
 & R is an upper triangular matrix

Ex :

$$A = \begin{bmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{bmatrix}$$

$$Q_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{Q_{12} A}_{A_1} = \begin{bmatrix} 3 & 27 & -4 \\ 0 & 20 & 14 \\ 4 & 11 & -2 \end{bmatrix}$$

$$c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{12}^2}} = 0$$

$$s = \frac{a_{12}}{\sqrt{a_{11}^2 + a_{12}^2}} = 1$$

$$Q_{13} = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

$$\underbrace{Q_{13} A_1}_{A_2} = \begin{bmatrix} 5 & 25 & -4 \\ 0 & 20 & 14 \\ 0 & -15 & 2 \end{bmatrix}$$

$$Q_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$Q_{23} A_2 = \begin{bmatrix} 5 & 25 & -4 \\ 0 & 25 & 10 \\ 0 & 0 & 10 \end{bmatrix}$$

R

$$\underbrace{Q_{23} Q_{13} Q_{12}}_{Q^T} A = R$$

Q is orthogonal

$$\Rightarrow A = QR$$

→ QR-Decomposition

- (i) For every $A \in \mathbb{R}^{n \times n}$, there exists an orthogonal matrix Q and an upper triangular matrix R s.t. $A = QR$.
- (ii) For a non-singular matrix $A \in \mathbb{R}^{n \times n}$ there exists a unique Q and a unique upper triangular matrix R with all positive diagonal entries s.t.
- $$A = QR$$

NL/02/09/25

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \longrightarrow \begin{bmatrix} \bigcirc \end{bmatrix} \begin{bmatrix} \cdot & \times & \times \\ \cdot & \times & \times \\ \bigcirc & \cdot & \times \end{bmatrix}$$

A Orthogonal R

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} \bigcirc \end{bmatrix} \begin{bmatrix} \cdot & \times & \times \\ \cdot & \times & \times \\ \cdot & \cdot & \times \end{bmatrix}$$

→ Another Approach to get an upper Δ^r matrix:

Assumptions:

(i) All leading principal submatrices of $A \in \mathbb{R}^{n \times n}$ are non-singular.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$A_1 = [*]$
 $A_2 = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$

k^{th} leading principal submatrix of A is A_k obtained by intersecting

1^{st} k rows & columns of A

for $k = 1, 2, \dots, n$

$\det(A_k) \neq 0$ $\forall$ all $k = 1, 2, \dots, n$

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

$\downarrow$ 0 $\downarrow$ 0

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} -\alpha_1^T & - \\ -\alpha_2^T & - \\ -\alpha_3^T & - \end{bmatrix}$$

A

$$= \alpha_1 \alpha_1^T + \alpha_2 \alpha_2^T + \alpha_3 \alpha_3^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

$\downarrow$ A

$$R_2 \leftarrow R_2 - 2R_1$$

$$\begin{matrix} R_2 \leftarrow R_2 - m_{21}R_1 \\ R_3 \leftarrow R_3 - m_{31}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 5/2 \end{bmatrix}$$

M_1 A $A^{(1)}$

$m_{21} = 2$
 $m_{31} = 1/2$

$$R_3 \leftarrow R_3 - m_{32}R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 5/2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{bmatrix}$$

M_2 $A^{(1)}$ $A^{(2)}$

$m_{32} = 1$

$$M_2 \left(\underbrace{M_1, A}_{A^{(1)}} \right) = A^2$$

$\underbrace{\hspace{10em}}_{A^{(2)}}$

$$M_2 M_1 A = A^{(2)} = U$$

↓
Lower Δ^n matrix
& it is invertible

$$A = \underbrace{(M_2 M_1)^{-1}}_{\text{Lower } \Delta^n \text{ matrix } L} U$$

$$\boxed{A = LU} \leftarrow \text{LU factorization}$$

$$\rightarrow A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{aligned} \det(A_1) &= 2 \\ \det(A_2) &= 2 \\ \det(A_3) &= \end{aligned}$$

$$\downarrow \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 5/2 \end{bmatrix}$$

$$\begin{aligned} R_2 &\leftarrow R_2 - (m_{21}) R_1 & m_{21} &= 2 \\ R_3 &\leftarrow R_3 - (m_{31}) R_1 & m_{31} &= 1/2 \end{aligned}$$

Gaussian Elimination
without row change

$$\downarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}$$

$$\begin{aligned} R_3 &\leftarrow R_3 - (m_{32}) R_2 \\ m_{32} &= 1 \end{aligned}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$$

$$A = LU$$

$$m_{i1} = \frac{a_{i1}}{a_{11}}$$

NL/03/09/25

$A \longrightarrow LU$ Decomposition
 $A = LU$

(All leading principal submatrices are non-singular)

$A \longrightarrow QR$ Decomposition
 $A = QR$

$L \rightarrow$ Lower Δ^n

$U \text{ or } R \rightarrow$ Upper Δ^n matrices

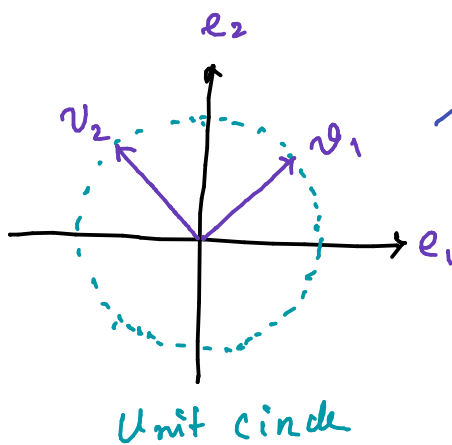
$Q \rightarrow$ Orthogonal matrix

$$\rightarrow A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

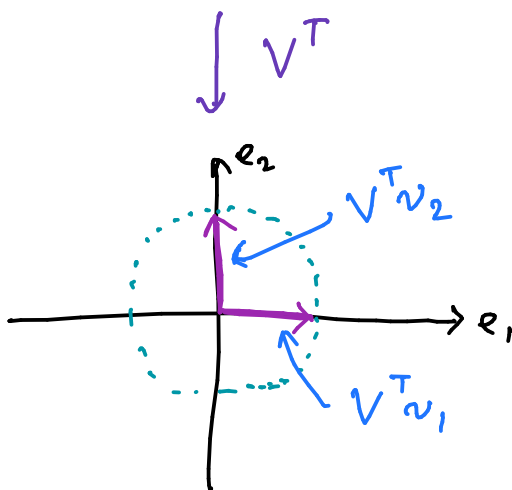
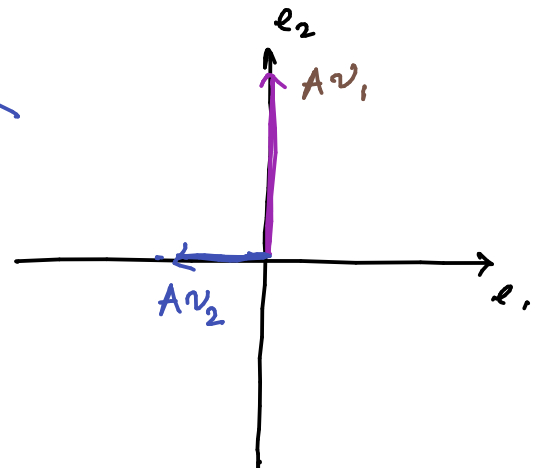
$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$$



A
 A is making magnification + Rotation



$$V = \begin{bmatrix} 1 & 1 \\ v_1 & v_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

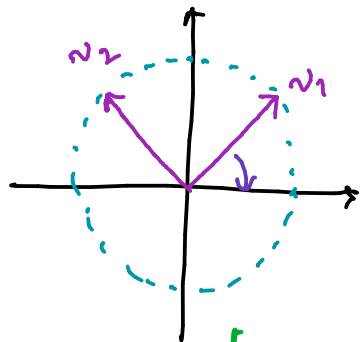
$$V^T = \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix}$$

$$V^T V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

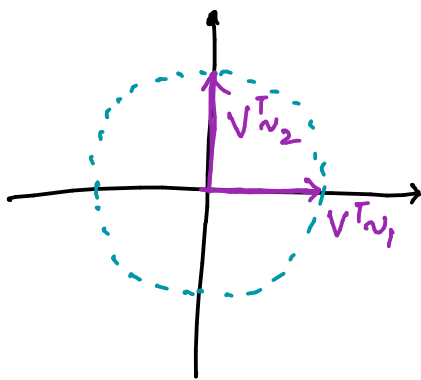
$$V^T v_1 = ?$$

$$V^T v_2 = ?$$

$$V^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \theta = 45^\circ$$



Rotation V^T



$$V^T v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V^T v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Magnification
along principal
axes through Σ

Magnification

$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\Sigma V^T v_1 = \Sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

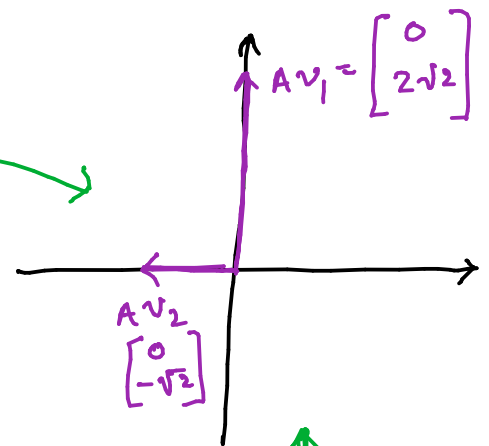
$$\Sigma V^T v_2 = \Sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$$A = U \Sigma V^T$$

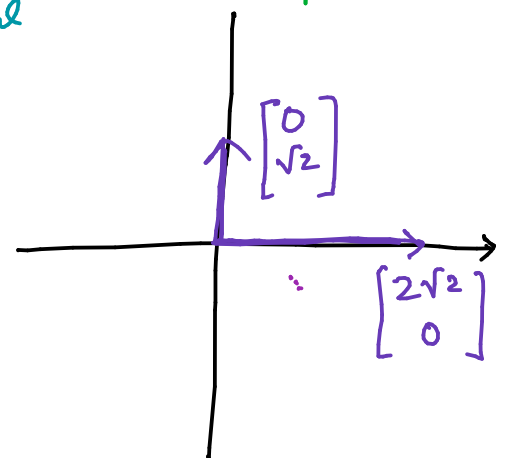
Singular Value
Decomposition
of A

U & V are orthogonal

Σ is a diagonal matrix
with non-negative entries
in diagonal.



Rotation U



→ Singular Value Decomposition

Let $A \in \mathbb{R}^{n \times m}$ matrix with rank $= r$
 $r \leq \min(n, m)$. Then, A can be expressed
as

$$A = U \Sigma V^T$$

where U & V are orthogonal matrices
and Σ is a non square matrix of
same size as A , & it has following form:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & \circ & & & \circ \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}_{n \times m}$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$$

σ_i : Singular values of A

u_i : right singular vector of A

v_i : Left singular vector of A

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}_{n \times n} \quad V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix}_{m \times m}$$

$$\left. \begin{aligned} A &= U \Sigma V^T \\ A^T &= V \Sigma^T U^T \end{aligned} \right\}$$

$\rightarrow$

$$AV = U\Sigma$$

$$A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \\ & & & 0 & \ddots & 0 \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

$$\vdots$$

$$Av_n = \sigma_n u_n$$

$$Av_{n+1} = 0$$

$$\vdots$$

$$Av_m = 0$$

All vectors
are orthonormal

NL/09/09/25

$$A \in \mathbb{R}^{n \times m}$$

$$A = U \Sigma V^T$$

where $U_{n \times n}$ is an orthogonal matrix
size $n \times n$ V is $m \times m$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r & & 0 \\ 0 & & & & 0 & \dots & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 \dots = 0$$

$$A = U \Sigma V^T$$

$$\rightarrow A^T = V \Sigma^T U^T$$

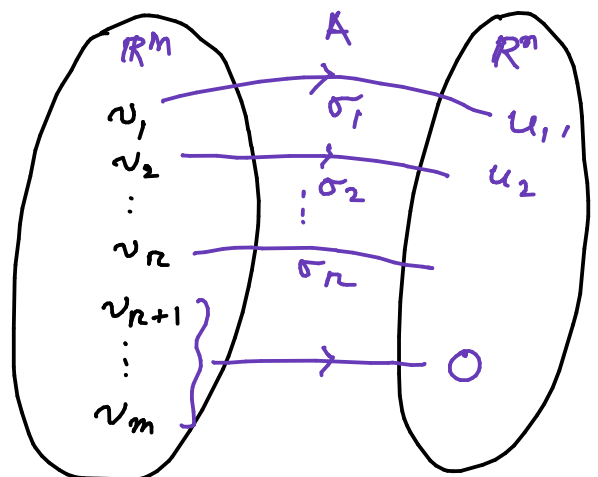
$$\Rightarrow A^T U = V \Sigma^T$$

$$A V = U \Sigma$$

$$A \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r & & 0 \\ 0 & & & & 0 & \dots & 0 \end{bmatrix}$$

$$\begin{aligned} \text{rank}(A) &= r \\ \Downarrow \\ \dim(\mathcal{R}(A)) &= r \end{aligned}$$

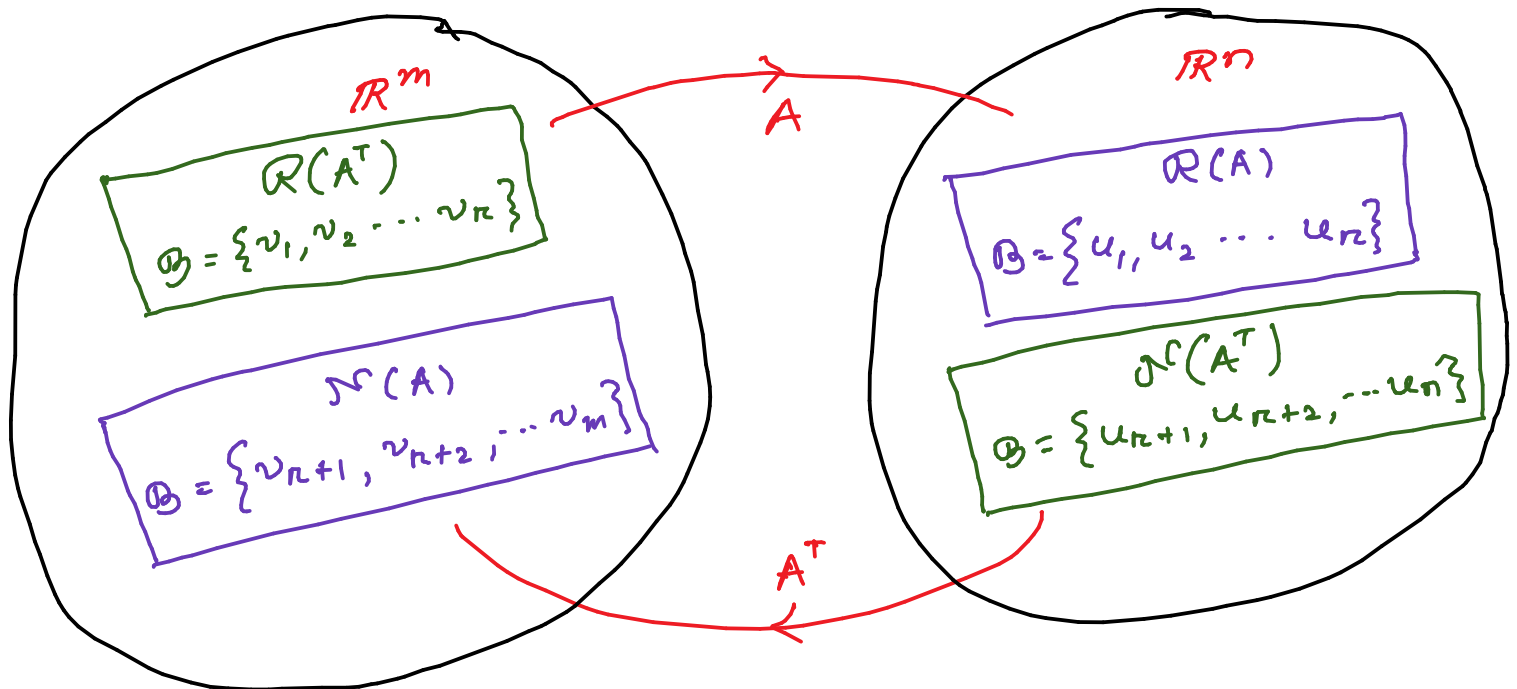
$$\left. \begin{aligned} A v_1 &= \sigma_1 u_1 \\ A v_2 &= \sigma_2 u_2 \\ &\vdots \\ A v_r &= \sigma_r u_r \\ A v_{r+1} &= 0 \\ &\vdots \\ A v_m &= 0 \end{aligned} \right\}$$



Given $A \in \mathbb{R}^{n \times m}$

with $\text{rank}(A) = r$

$$\begin{array}{ccccc}
 & \overset{A}{\sigma_1} & & \overset{A^T}{\sigma_1} & \\
 v_1 & \xrightarrow{\quad} & u_1 & \xrightarrow{\quad} & v_1 \\
 v_2 & \xrightarrow{\quad \sigma_2 \quad} & u_2 & \xrightarrow{\quad \sigma_2 \quad} & v_2 \\
 \vdots & \vdots & \vdots & \vdots & \\
 v_r & \xrightarrow{\quad \sigma_r \quad} & u_r & \xrightarrow{\quad \sigma_r \quad} & v_r \\
 \left. \begin{array}{l} v_{r+1} \\ \vdots \\ v_m \end{array} \right\} & \longrightarrow & 0 & \left. \begin{array}{l} u_{r+1} \\ \vdots \\ u_n \end{array} \right\} & \longrightarrow 0
 \end{array}$$



→ SVD Computation

$$\text{Let } A = U \Sigma V^T$$

$$\begin{aligned}
 \Rightarrow A^T A &= V \Sigma^T U^T U \Sigma V^T \\
 &= V \tilde{\Sigma}^2 V^T
 \end{aligned}$$

↑
Notation

$$A^T = V \Sigma^T U^T$$

$$\begin{aligned}
 A A^T &= U \Sigma V^T V^T \Sigma^T U^T \\
 &= U \tilde{\Sigma}^2 U^T
 \end{aligned}$$

↑
Notation

$$\bar{\Sigma}^2 = \Sigma^T \Sigma$$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n & & \\ & & & & 0 & \ddots & 0 \end{bmatrix}_{m \times n} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n & & \\ & & & & 0 & \ddots & 0 \end{bmatrix}_{n \times m}$$

$$= \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 & & \\ & & & & 0 & \ddots & 0 \end{bmatrix}_{m \times m}$$

$$A^T A \rightarrow m \times m$$

$$A^T A = V \bar{\Sigma}^2 V^T$$

$$\Rightarrow V^T (A^T A) V = \bar{\Sigma}^2$$



Perform Eigenvalues & Eigenvectors computation.

$$\text{def } A^T A \xrightarrow{\text{non-zero}} \text{Eigenvalues}$$



$$\lambda_1(A^T A), \lambda_2(A^T A), \dots, \lambda_n(A^T A)$$

$$\sigma_1^2 = \lambda_1(A^T A)$$

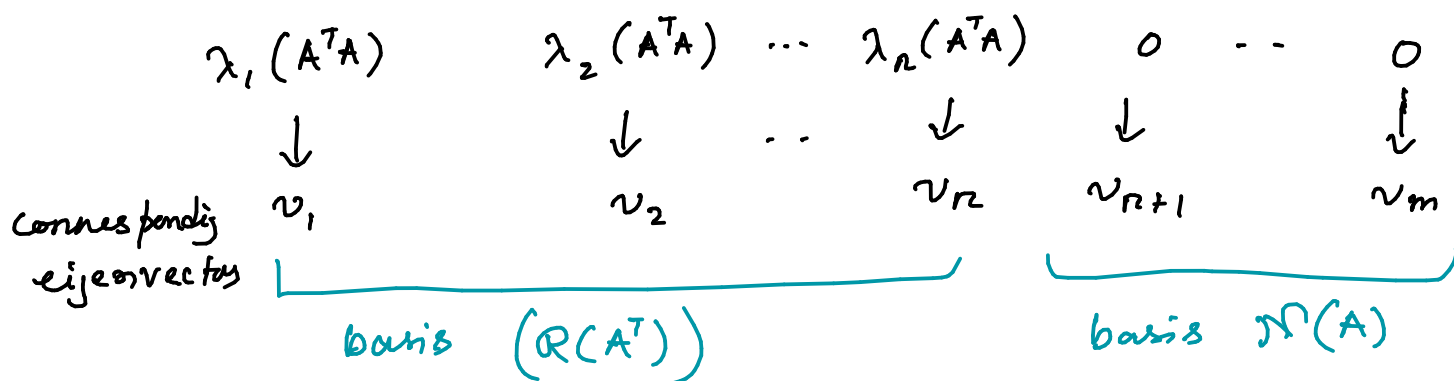
$$\sigma_2^2 = \lambda_2(A^T A)$$

⋮

$$\sigma_n = \lambda_n(A^T A)$$

$$\Rightarrow \sigma_i = \sqrt{\lambda_i(A^T A)}$$

↑
They are all non-negative numbers



→ The eigenvectors of a symmetric matrix
are orthogonal.

$$A = A^T$$

$\downarrow$

Eigenvalues

$$\lambda_i(A)$$

$$\lambda_j(A)$$

Eigenvectors

$$\downarrow$$

$$v_i$$

$$\downarrow$$

$$v_j$$

$$i \neq j$$

$$\lambda_i \neq \lambda_j$$

$$A v_i = \lambda_i v_i$$

$$A v_j = \lambda_j v_j$$

$$\lambda_j v_i^T v_j = v_i^T (\lambda_j v_j)$$

$$= v_i^T (A v_j)$$

$$= (v_i^T A^T) v_j \quad \text{since } A = A^T$$

$$= \lambda_i v_i^T v_j$$

$$\Rightarrow \lambda_j v_i^T v_j - \lambda_i v_i^T v_j = 0$$

$$\Rightarrow (\lambda_j - \lambda_i) v_i^T v_j = 0$$

$$\text{Since } \lambda_i \neq \lambda_j \Rightarrow v_i^T v_j = 0 \Rightarrow v_i \perp v_j$$

$$A A^T = U \tilde{\Sigma}^2 U^T$$

$$\Rightarrow U^T (A A^T) U = \tilde{\Sigma}^2$$

$$\tilde{\Sigma}^2 = \Sigma \Sigma^T = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}_{n \times n}$$

Perfor eigenvalue & eigenvector computation

Eigenvalues	$\lambda_1(AA^T)$	$\lambda_2(AA^T)$	$\dots$	$\lambda_n(AA^T)$	0	$\dots$	0
	$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$	$\dots$	$\downarrow$
Eigenvectors	u_1	u_2		u_n	u_{n+1}		u_n
	basis $R(A)$				basis $N(A^T)$		

NL/10/09/25

• Induced matrix 2-norm

$$\|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

Let SVD of $A \in \mathbb{R}^{n \times m}$ be $A = U \Sigma V^T$

with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 \dots 0$

$\text{rank}(A) = r$

$$\boxed{\|A\|_2 = \sigma_1}$$

σ_1 : Largest singular value of A .

→ We need to show that

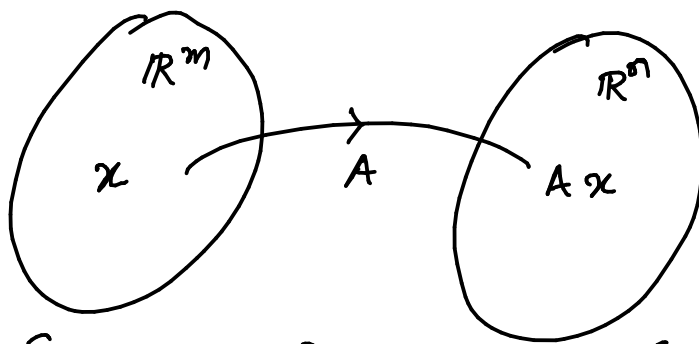
$$\max_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1 \quad \text{for } x \neq 0$$

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix} \quad U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}$$

$$\frac{\|Av_1\|_2}{\|v_1\|_2} = \frac{\|\sigma_1 u_1\|_2}{\|v_1\|_2} = \sigma_1 \quad \|u_1\|_2 = \|v_1\|_2 = 1$$

Let us choose an x

$$\text{s.t.} \quad \max_x \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_1 \quad \dots \quad (1)$$



$$A \in \mathbb{R}^{n \times m}$$

$$B = \{v_1, v_2, \dots, v_m\}$$

$$B = \{u_1, u_2, \dots, u_n\}$$

For any $x \in \mathbb{R}^m$ $x \neq 0$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix}}_V \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}}_\alpha$$

$$\|Ax\|_2 \leq \|A\| \|x\|_2$$

$$\|x\|_2^2 = \|V\alpha\|_2^2 = \|\alpha\|_2^2 \quad \text{since } V \text{ is orthogonal}$$

$$= \alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2 \quad \dots \quad \textcircled{xx}$$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$$

$$Ax = \alpha_1 Av_1 + \alpha_2 Av_2 + \dots + \alpha_m Av_m$$

$$= \alpha_1 \sigma_1 u_1 + \alpha_2 \sigma_2 u_2 + \dots + \alpha_n \sigma_n u_n + 0 + \dots + 0$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}}_{\tilde{U}} \underbrace{\begin{bmatrix} \alpha_1 \sigma_1 \\ \alpha_2 \sigma_2 \\ \vdots \\ \alpha_n \sigma_n \end{bmatrix}}_{\alpha_\sigma}$$

$$Ax = \tilde{U} \alpha_\sigma$$

$$\|Ax\|_2^2 = \|\tilde{U} \alpha_\sigma\|_2^2$$

$$= \|\alpha_\sigma\|_2^2$$

Since $\tilde{U}$ is orthogonal

$$= \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + \dots + \alpha_n^2 \sigma_n^2$$

$$\text{Since } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \left(\leq \sigma_1^2 (\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2) \right)$$

$$\leq \sigma_1^2 (\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2)$$

$$= \sigma_1^2 \|x\|_2^2 \quad \dots \text{ from } \textcircled{**}$$

$$\|Ax\|_2 \leq \sigma_1 \|x\|_2$$

$$\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_1 \quad \text{for every } x \in \mathbb{R}^m.$$

$\textcircled{2}$

In $\textcircled{1}$ we have chosen x

$$\text{s.t.} \quad \max_x \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_1$$

This says that the ratio $\frac{\|Ax\|_2}{\|x\|_2}$ will always be $\leq \sigma_1$ whatever x you choose.

$$\text{This implies } \rightarrow \max_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$

$\Downarrow$

$$\|A\|_2 = \sigma_1$$

□

$$\rightarrow \|A\|_2 = \|A^T\|_2 = \sigma_1$$