

NL/19/09/25

$X \quad Y \leftarrow \text{subspaces}$

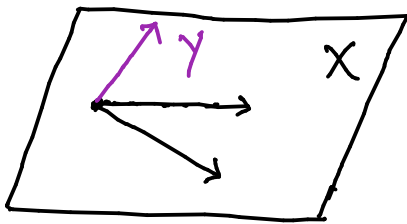
$$V = X + Y := \{x + y \mid x \in X \text{ and } y \in Y\}$$

↑
Addition of
two subspaces



There would be
some element
which belongs to

$$X \cap Y$$

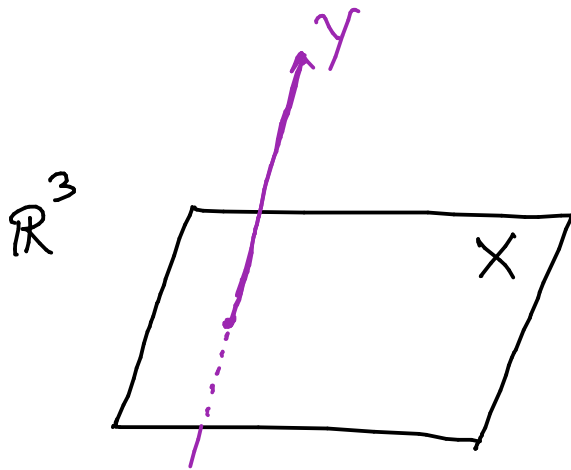


$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Y

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

X



$$Y = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

↑
The intersection

element $X \cap Y = 0$

↑
zero-vector

In subspace addition

(i) $X + Y$ with non-zero elements
in $X \cap Y$

(ii) $X + Y$ with only '0' vector in
 $X \cap Y$

X & Y are called "Complementary
subspaces".
of a
vector space V .

$V = X \oplus Y$ where X & Y are
complementary

V is "direct sum" of two subspaces
 X & Y

→ For a vector space V with
subspaces X & Y having bases as

$$X \rightarrow B_X = \{x_1, x_2 \dots x_r\}$$

$$Y \rightarrow B_Y = \{y_1, y_2 \dots y_k\}$$

$$\text{s.t. } r + k = n$$

Following statements are equivalent:

(i) $V = X \oplus Y$

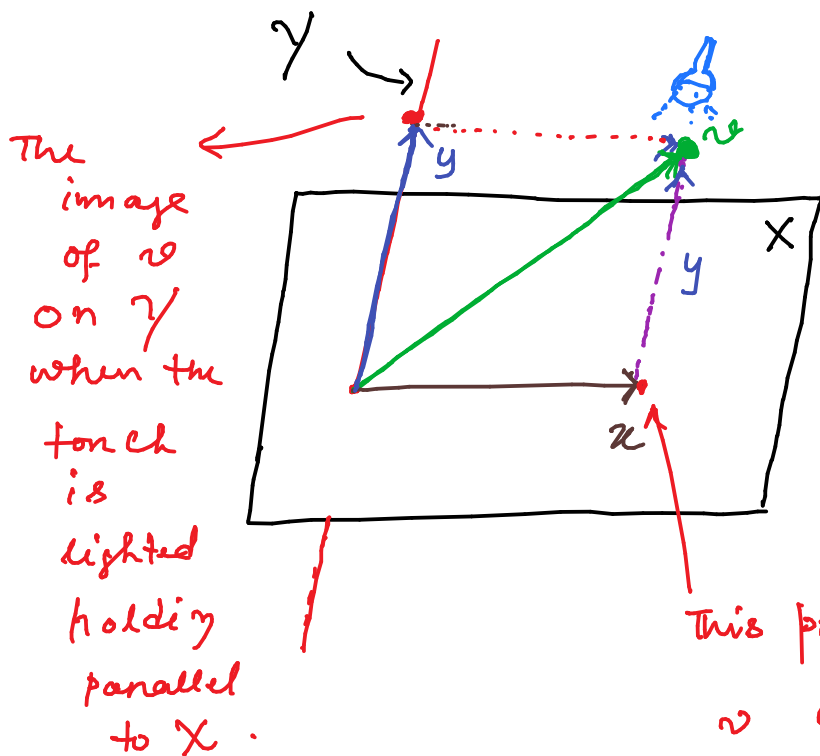
(ii) $B_X \cap B_Y = \emptyset$

$\& B_X \cup B_Y$ is a basis for V .

(iii) For each vector $v \in V$ there are unique two vectors $x \in X$ $\& y \in Y$

s.t. $v = x + y$.

$\uparrow$ $\uparrow$
X-component of v Y-component of v



'x' is projection of v onto X along Y .

This point is the image of v when the torch is lighted along the direction of Y

→ Projection

Let $V = X \oplus Y$ so that there are two unique vectors $x \in X$ & $y \in Y$ such that for every $v \in V$

$$v = x + y. \quad \text{Then,}$$

- (i) the vector x is called "projection" of v onto X along Y .
- (ii) the vector y is called "projection" of v onto Y along X .

→ The Projection P

$$V = \mathbb{R}^n \quad \boxed{Pv = x} \leftarrow P \text{ must have this property.}$$

$$\text{For } X \rightarrow \text{basis} \rightarrow B_X = \{x_1, x_2, \dots, x_n\}$$

$$\text{For } Y \rightarrow \text{basis} \rightarrow B_Y = \{y_1, y_2, \dots, y_{n-n}\}$$

$$\text{Since } V = X \oplus Y \Rightarrow B_X \cap B_Y = \emptyset$$

Construct

$$B = \left[\underbrace{x_1 \ x_2 \ \dots \ x_n}_{X_m} \mid \underbrace{y_1 \ y_2 \ \dots \ y_{n-n}}_{Y_m} \right]_{n \times n}$$

$$x_i \in \mathbb{R}^n$$

$$y_i \in \mathbb{R}^n$$

All vectors are
linearly independent

B is
non-singular
matrix

$$\text{Sim } B_x \cap B_y = \emptyset \text{ empty}$$

$$PB = P \left[\underbrace{x_1 \ x_2 \ \dots \ x_n}_{X_m} \mid \underbrace{y_1 \ y_2 \ \dots \ y_{n-n}}_{Y_m} \right]$$

$$= P[X_m \mid Y_m] = [PX_m \mid PY_m]$$

P is a projection matrix s.t. $Pv = x$
onto X along Y

$$v = x + y$$

$$x \in X$$

$$y \in Y$$

$$\Downarrow$$

$$P(x+y) = x$$

$$\Downarrow$$

$$\begin{cases} Px = x & \dots \textcircled{1} \\ Py = 0 & \dots \textcircled{2} \end{cases}$$

$$\begin{cases} Px_i = x_i & \text{for } i = 1, 2, \dots, n \\ Py_i = 0 & \text{for } i = 1, 2, \dots, n-n \end{cases}$$

$$PB = P \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_{n-n} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= [X_m; 0]$$

$$P = [X_m; 0] B^{-1} \quad \leftarrow \text{Projection which projects } v \text{ onto } X \text{ along } Y.$$

↓

$$X = B \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

$$P = B \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} B^{-1}$$

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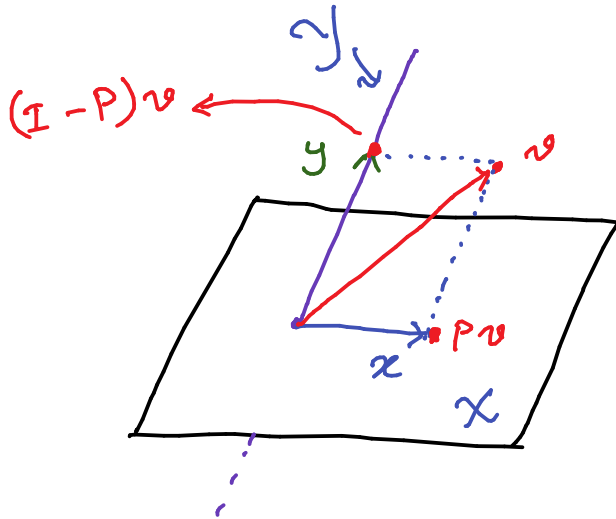
Complementary subspaces

$$X \cap Y = \{0\}$$

$$v \in \mathbb{R}^n \rightarrow v = x + y$$

with $x \in X$

$y \in Y$



The x is 'project' of v

onto X along Y

y is 'project' of v

onto Y along X

The associated Projector P
onto X is

$$P = \underbrace{\begin{bmatrix} X_m & Y_m \end{bmatrix}}_B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} X_m & Y_m \end{bmatrix}^{-1}}_B$$

$$= \begin{bmatrix} X_m & 0 \end{bmatrix} \begin{bmatrix} X_m & Y_m \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} X_m & 0 \end{bmatrix} B^{-1}$$

$$P x = x$$

$$\underline{x} = P v$$

$$= [X : 0] \underbrace{B^{-1} v}_{z \in \mathbb{R}^n}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ z_{n+r} \\ \vdots \\ z_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_1 & x_2 & \dots & x_n & 0 & \dots & 0 \end{bmatrix}}_{\in X} z$$

→ Complementary Projection

$$\boxed{I_n - P}$$

$$I - P = B B^{-1} - B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} B^{-1}$$

$$= B \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} B^{-1}$$

$$(I - P) v = B \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \underbrace{B^{-1} v}_{z \in \mathbb{R}^n}$$

$$= B \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} z$$

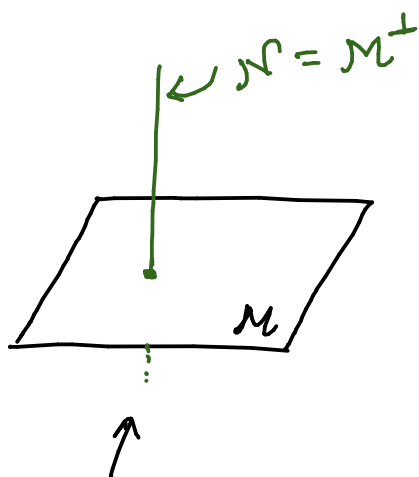
$$= \begin{bmatrix} x_m : y_m \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} z = \begin{bmatrix} 0 : y_m \end{bmatrix} z$$

$$\Rightarrow (I - P)v = \underbrace{[0 : \gamma_m]}_{\in \mathcal{Y}} \dot{z}$$

$$\boxed{\begin{aligned} y &= (I - P)v \\ x &= Pv \end{aligned}}$$

- $\mathcal{R}(P) = \{x \mid Px = x\}$
- $\mathcal{R}(P) = \mathcal{N}(I - P) = \mathcal{X}$
- $\mathcal{R}(I - P) = \mathcal{N}(P) = \mathcal{Y}$
 $\uparrow$
 Image space

$\rightarrow$



$$\mathcal{B}_M = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{B}_N = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Every vector in $\mathcal{M}$
 are orthogonal to
 every vector in $\mathcal{N}$

For a set $M \subset \mathbb{R}^n$, the
"Orthogonal Complement" of M ,

denoted as $M^\perp$, is defined as follows:

$$M^\perp = \left\{ x \in \mathbb{R}^n \mid m^T x = 0, \forall m \in M \right\}$$



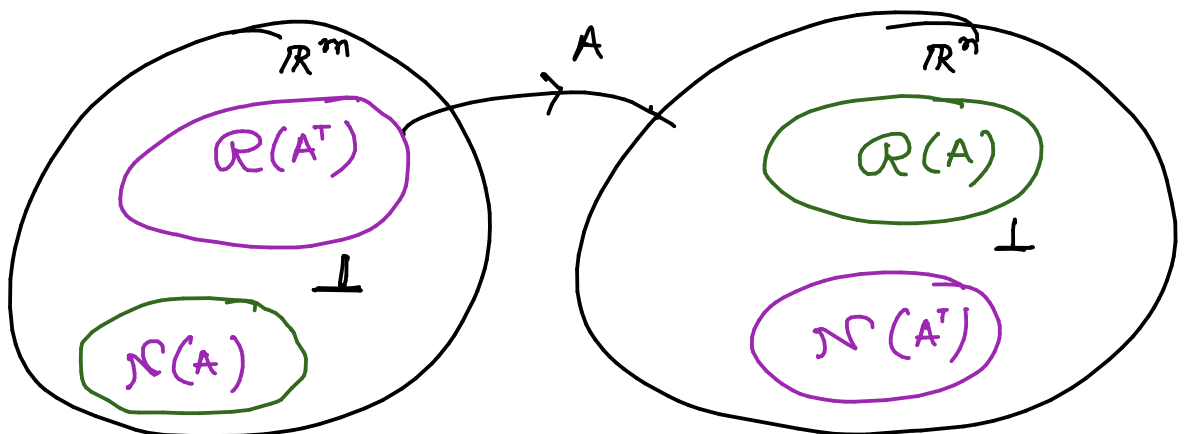
Orthogonal complement of M .

- $M^\perp$ is a subspace
- $\dim(M^\perp) = n - \dim(M)$
- $(M^\perp)^\perp = M$

$$\mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathcal{U} = M \oplus M^\perp$$

•

$$A \in \mathbb{R}^{n \times m}$$



- Fact $[\mathcal{R}(A)]^\perp = \mathcal{N}(A^T)$

Proof

Let z be a vector in the range of A

$$z \in \mathcal{R}(A)$$

$\Downarrow$

There exists a vector $y \neq 0$, $y \in \mathbb{R}^m$

s.t. $Ay = z$

For $z \in [\mathcal{R}(A)]^\perp$

$\Uparrow$

$$z^T z = 0 \quad z \in \mathcal{R}(A)$$

$\Uparrow$

$$y^T A^T z = 0$$

$\Uparrow$

$$A^T z = 0$$

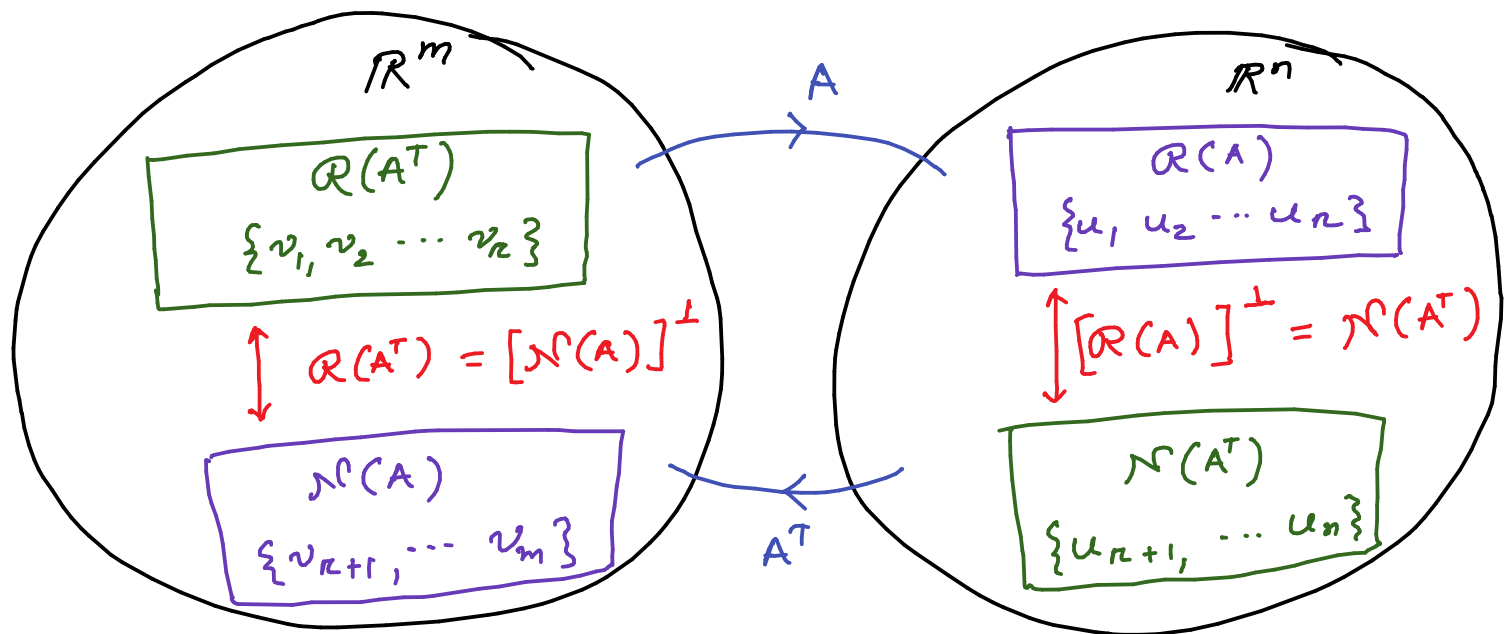
$\Uparrow$

$$z \in \mathcal{N}(A^T)$$

- $[\mathcal{N}(A)]^\perp = \mathcal{R}(A^T)$

- $$\begin{aligned} \mathbb{R}^n &= \mathcal{R}(A) \oplus [\mathcal{R}(A)]^\perp \\ &= \mathcal{R}(A) \oplus \mathcal{N}(A^T) \end{aligned}$$

$$\begin{aligned} \bullet \quad \mathbb{R}^m &= \mathcal{N}(A) \oplus [\mathcal{N}(A^T)]^\perp \\ &= \mathcal{N}(A) \oplus \mathcal{R}(A^T) \end{aligned}$$



$$\begin{aligned} \mathbb{R}^m &= \mathcal{N}(A) \oplus [\mathcal{N}(A)]^\perp \\ &= \mathcal{N}(A) \oplus \mathcal{R}(A^T) \end{aligned}$$

$$\begin{aligned} \mathbb{R}^n &= \mathcal{R}(A) \oplus [\mathcal{R}(A)]^\perp \\ &= \mathcal{R}(A) \oplus \mathcal{N}(A^T) \end{aligned}$$

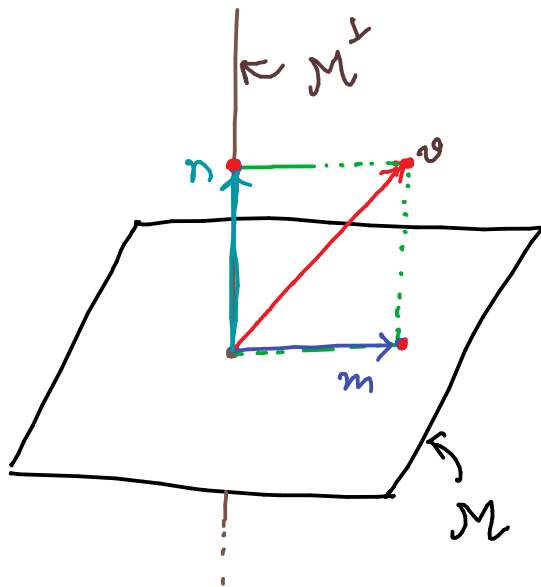
$$A \in \mathbb{R}^{n \times m}$$

$$A = U \Sigma V^T \quad \text{with} \quad \text{rank}(A) = r$$

$$U = [u_1 \ u_2 \ \dots \ u_r \ : \ u_{r+1} \ \dots \ u_n]_{n \times n}$$

$$V = [v_1 \ v_2 \ \dots \ v_r \ : \ v_{r+1} \ \dots \ v_m]_{m \times m}$$

→ The Orthogonal Projection



$$m = P_M v$$

$$n = (I - P_M) v$$

$$v = m + n$$

$$\text{with } m \in M$$

$$n \in M^\perp$$

m : The projection of v onto M along $M^\perp$

↑
called "Orthogonal Projection of v "

n : The projection of v onto $M^\perp$ along M .

- The associated projectors are called "Orthogonal Projectors".
 $\swarrow$
 P_M

$$\mathcal{M} = \mathcal{B}_{\mathcal{M}} = \{m_1, m_2, \dots, m_n\} \leftrightarrow \underline{\dim(\mathcal{M}) = n}$$

$$\mathcal{M}^\perp = \mathcal{B}_{\mathcal{M}^\perp} = \{n_1, n_2, \dots, n_k\} \quad \text{s.t.} \quad \underline{k+n = n}$$

Matrix

$$B = \left[\underbrace{m_1 \ m_2 \ \dots \ m_n}_M \ \vdots \ \underbrace{n_1 \ n_2 \ \dots \ n_k}_N \right]$$

$$B = [M : N]$$

The projection

$$P_{\mathcal{M}} = B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} B^{-1} \quad \checkmark$$

$$= [M : 0] [M : N]^{-1}$$

→ let the columns of N be orthonormal.

$$\begin{bmatrix} (M^T M)^{-1} M^T \\ N^T \end{bmatrix} [M : N] = \begin{bmatrix} \underline{(M^T M)^{-1} M^T M} & (M^T M)^{-1} M^T N \\ N^T M & N^T N \end{bmatrix}$$

$$N^T N = I$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$M^T N = 0$$

$$N^T M = 0$$

$$BA = I \Rightarrow B = A^{-1}$$

$$[M : N]^{-1} = \begin{bmatrix} (M^T M)^{-1} M^T \\ \vdots \\ N^T \end{bmatrix}$$

Then orthogonal projection on M is

$$P_M = [M : 0] [M : N]^{-1}$$

$$= [M : 0] \begin{bmatrix} (M^T M)^{-1} M^T \\ N^T \end{bmatrix}$$

By assuming that
the basis of $N = M^\perp$
are orthonormal

$$P_M = M (M^T M)^{-1} M^T$$

The orthogonal
projection onto
subspace M .



if it is assumed that
the basis for M is
also orthonormal

↓ $M^T M = I$

$$P_M = M M^T$$

→ Let M be a subspace of $\mathbb{R}^n$ with dimension $\dim(M) = r$. Let the columns of matrices $M \in \mathbb{R}^{n \times r}$ form a basis for $M \subseteq \mathbb{R}^n$ subspaces. Then,

(i) The orthogonal projection onto subspace M is

$$P_M = M(M^T M)^{-1} M^T \quad (\text{assumption that the col. of } M \text{ are orthonormal})$$

(ii) The orthogonal projection onto subspace $M^\perp$

$$\text{is } P_{M^\perp} = N(N^T N)^{-1} N^T \quad (\text{assump that the col of } N \text{ are orthonormal})$$

(iii) Complementary Projectors

$$P_M \longrightarrow I - P_{M^\perp}$$

$$P_{M^\perp} \longrightarrow I - P_M$$

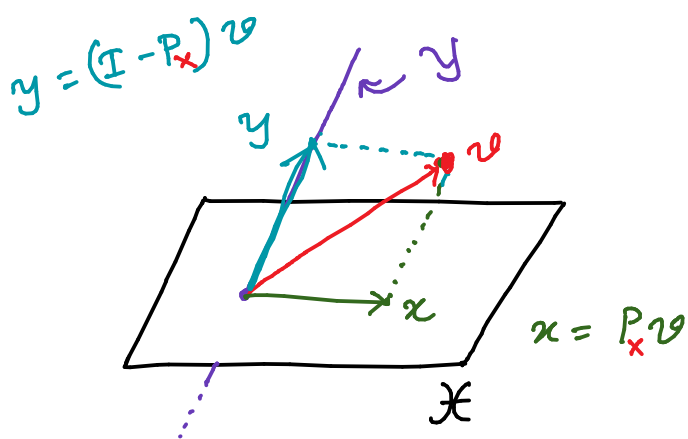
- If both matrices $M \in \mathbb{R}^{n \times k}$ and $N \in \mathbb{R}^{n \times r}$ columns are orthonormal (i.e. the orthonormal basis for $M \in \mathbb{R}^n$ and $N \in \mathbb{R}^n$) then the projection

$$P_M = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$\text{where } U = \begin{bmatrix} m_1 & m_2 & \dots & m_k & n_1 & n_2 & \dots & n_r \end{bmatrix}$$

$$k + r = n$$

- $P_M = MM^T$
- $P_{M^\perp} = NN^T$

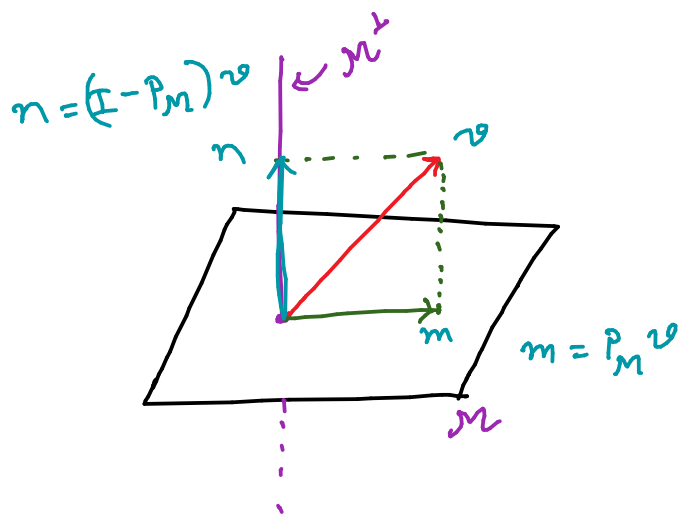


$$P_x = [x : 0][x : y]^{-1}$$

Oblique Projector

$x \rightarrow$ basis for X

$y \rightarrow$ basis for Y



$$P_M = [M : 0][M : N]^{-1}$$

$\downarrow$ simplify

$$P_M = M(M^T M)^{-1} M^T$$

$\uparrow$

Orthogonal Projector

→ Moore-Penrose Inverse

Pseudo-Inverse of a matrix

- A matrix A^+ is said to be pseudo-inverse of A if it satisfies following 4 properties:

$$(i) \quad A A^+ A = A$$

$$(ii) \quad A^+ A A^+ = A^+$$

$$(iii) \quad (A A^+)^T = A A^+$$

$$(iv) \quad (A^+ A)^T = A^+ A$$

the orthogonal $P_m = M \underbrace{(M^T M)^{-1} M^T}$

- If A is full column rank, then

$$A^+ = (A^T A)^{-1} A^T$$

$$\begin{bmatrix} M \end{bmatrix}_{n \times n}$$

- If A is full row rank, then

$$A^+ = A^T (A A^T)^{-1}$$

$$\begin{bmatrix} \end{bmatrix}$$

- The orthogonal projection onto subspace $\mathcal{M}$ (along $\mathcal{M}^\perp$) is

$$P_{\mathcal{M}} = M \underbrace{(M^T M)^{-1}}_{M^\dagger} M^T$$

$$P_{\mathcal{M}} = M M^\dagger$$

→ Computation of pseudo-inverse of A using SVD

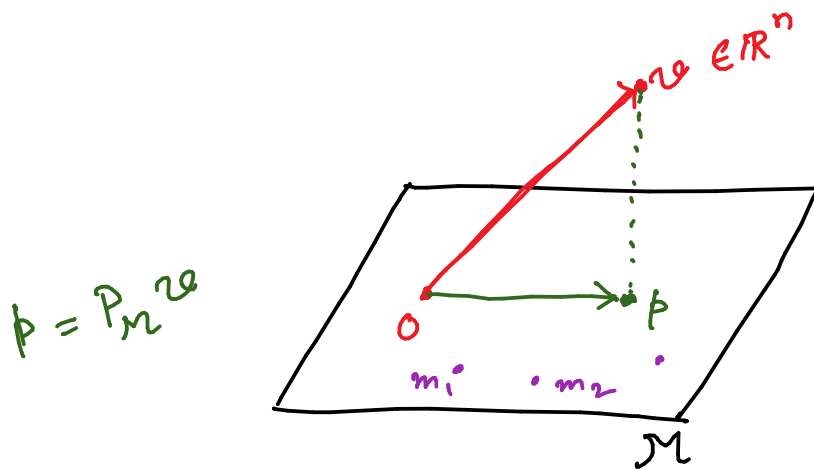
$$A \rightarrow U \Sigma V^T \quad \text{rank}(A) = r$$

$$A = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

- $AA^\dagger$ is a projection onto $\text{Im}(A)$.
- $A^\dagger A$ is a projection onto $\text{Im}(A^T)$

→



M is a
subspace of $\mathbb{R}^n$.

v is a point which does not belong
to M .

- The closest point between v & the
points in M is p , which is orthogonal
projection of v onto M

$$\text{dist}(v, M) := \min_{m \in M} \|v - m\|_2$$

$$= \|v - P_M v\|_2$$

$$= \|v - p\|_2$$