

NL/08/10/25

A set of m algebraic equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$\vdots$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Given $\{a_{ij}, b_i\}$

We need to solve for $\{x_i\}$

$$\begin{array}{lll} \rightarrow & x_1 + 2x_2 = 3 & \dots E_1 \\ & 4x_1 + 5x_2 = 6 & \dots E_2 \\ & \downarrow \text{Solve for } x_1 \text{ \& } x_2 & \end{array} \quad \begin{array}{ll} a_{11} = 1 & b_1 = 3 \\ a_{12} = 2 & b_2 = 6 \\ a_{21} = 4 & \\ a_{22} = 5 & \end{array}$$

One method
is to make it a triangular system

• Elimination:

$$\begin{aligned} E_2 - 4(E_1) &\rightarrow -3x_2 = -6 \\ &\Rightarrow x_2 = \underline{2} \end{aligned}$$

$$\bullet \text{ Back substitution } \rightarrow x_1 + 2 \cdot 2 = 3 \Rightarrow x_1 = \underline{-1}$$

$$x_1 + 2x_2 = 3$$

$$4x_1 + 5x_2 = 6$$

↓

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \end{bmatrix} x_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

↓

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}_b$$

→ soln x by
taking inverse
of A .

2nd method

↓

→

$$x_1 + 2x_2 = 3 \quad \dots E_1$$

$$4x_1 + 8x_2 = 12 \quad \dots E_2$$

↓

soln

$$E_{\text{elimination}}: E_2 - 4E_1 \rightarrow \underline{0 = 0}$$

↓

$$x_1 + 2x_2 = 3$$

↑
One equation
is redundant

$$\{x_1 = 1, x_2 = 1\}, \{x_1 = 2, x_2 = 0.5\}$$

$$\{x_1 = 3, x_2 = 0\}$$

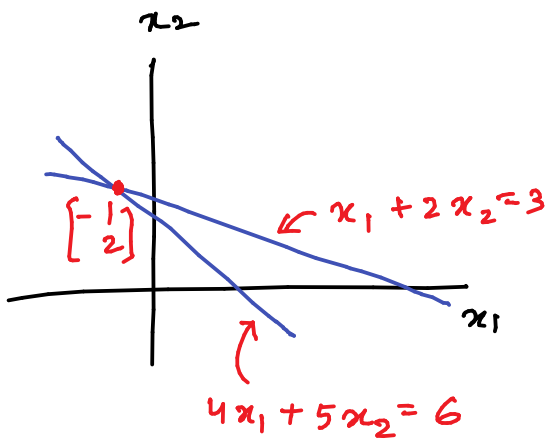
↑

Infinitely many solutions

$$\rightarrow \begin{aligned} x_1 + 2x_2 &= 3 \quad \dots E_1 \\ 4x_1 + 8x_2 &= 6 \quad \dots E_2 \end{aligned}$$

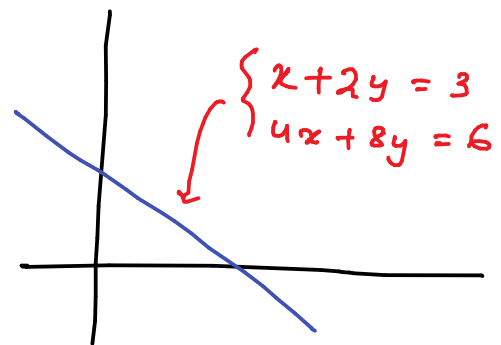
$$E \text{ eliminated} \rightarrow E_2 - 4E_1 \rightarrow \underline{0 = -6}$$

$\uparrow$
 Inconsistent E_2
 $\uparrow$
 No Solutions.



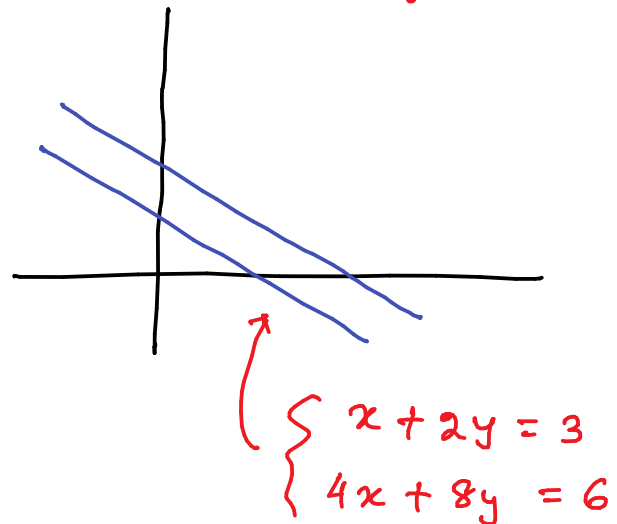
One Solution

Non-Singular
Case



Infinitely many solⁿ.

Singular Case



No solution

Singular Case

For non-singular case $\rightarrow$ Only one Solⁿ

Singular cases $\begin{cases} \rightarrow \text{No solution} \\ \rightarrow \text{Infinitely many solution.} \end{cases}$

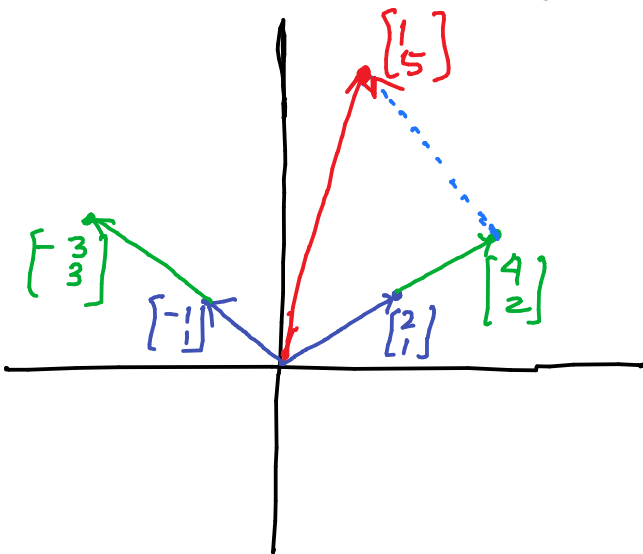
$\rightarrow$ Geometric View :

$$\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 5 \end{cases} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2 = \underbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}_b$$

$\rightarrow$ Solution $x_1 = 2$
 $x_2 = 3$

Linear combination
of two vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ producing } \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

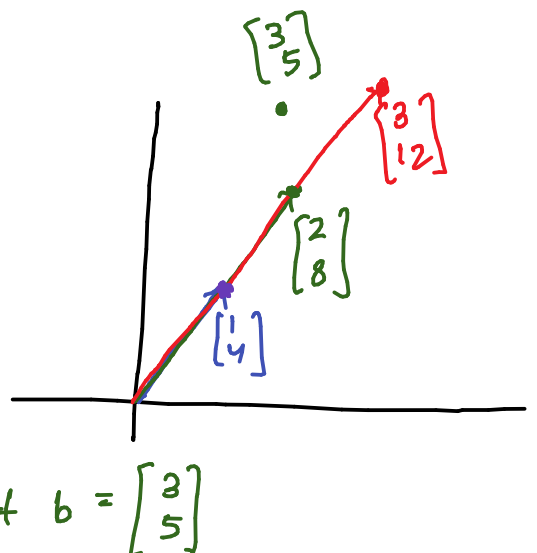


$$\begin{cases} x_1 + 2x_2 = 3 \\ 4x_1 + 8x_2 = 12 \end{cases}$$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 8 \end{bmatrix} x_2 = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

$\rightarrow$ Different $b = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$



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System of linear equations:

$$\left\{ \begin{array}{l} m \text{ algebraic equations} \\ n \text{ variables} \end{array} \right.$$

Singular systems $\left\{ \begin{array}{l} \cdot \text{ Unique solution} \} \text{ Non singular} \\ \cdot \text{ No solution} \\ \cdot \text{ Infinitely many sol}^n. \end{array} \right.$

$$\left. \begin{array}{l} x_1 + x_2 + 2x_3 = 3 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 4 \end{array} \right\} \begin{array}{l} 3 \text{ equations \& 3 unknowns.} \\ \text{Square system} \end{array}$$

↓

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{a_1} x_1 + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{a_2} x_2 + \underbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}_{a_3} x_3 = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}}_b \rightarrow b \text{ is obtained taking linear combination of } a_1, a_2, a_3$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}}_b$$

$$Ax = b$$

A & b are given
solve for x.

- Construct an augmented matrix $[A:b]$

• Case-I

$$\begin{array}{c} \text{Pivot Position} \rightarrow \\ \begin{bmatrix} \textcircled{1} & 1 & 2 & \vdots & 3 \\ 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 3 & \vdots & 4 \end{bmatrix} \\ \begin{array}{l} 0 \leftarrow \\ 0 \leftarrow \end{array} \end{array}$$

The element in a pivot position is called "pivot element".

$$\downarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & \textcircled{0} & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} \textcircled{1} & 1 & 2 & \vdots & 3 \\ 0 & \textcircled{1} & 1 & \vdots & 2 \\ 0 & 0 & \textcircled{1} & \vdots & 2 \end{bmatrix}$$

$\tilde{A}$ $\downarrow$ $\tilde{c}$

$$x_3 = 2$$

$$x_2 + x_3 = 2 \Rightarrow x_2 = 0$$

$$x_1 + x_2 + 2x_3 = 3 \Rightarrow x_1 = -1$$

For square case $\rightarrow A$ is square

- we consider $(i,i)^{th}$ position as pivot position

- A pivot element must be non-zero.

The above elimination process is called

"Gauss Elimination Process".

$$\begin{matrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} x_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \\ a_1 \qquad \qquad a_2 \qquad \qquad a_3 \qquad \qquad b \end{matrix}$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

Are $\{a_1, a_2, a_3\}$ linearly independent?

• "The number of pivot elements in $\tilde{A}$ is equal to the number of linearly independent columns present in A ."

↓

• 'b' is obtained by taking linear combinations of the columns of A . If $A \in \mathbb{R}^{n \times n}$ & $b \in \mathbb{R}^n$, then one can always solve for b iff all the columns span n -dimensional space.

↑

called "Non-singular System"

(A is non-singular)

→ Case - II

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{array} \right]$$

$\underbrace{\hspace{10em}}_A$
 $\underbrace{\hspace{2em}}_b$

↓

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 3 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

← infinitely many solⁿ.

← Singular Systems

Logically true

⇒ It has a solution

2 - linearly independent columns in A
 Since 2 - pivot elements.

$$x_2 + x_3 = 2$$

$$\Rightarrow x_2 = 2 - x_3 \quad \rightarrow \text{free variable.}$$

$$\text{Let } x_3 = 0 \quad \Rightarrow x_2 = 2$$

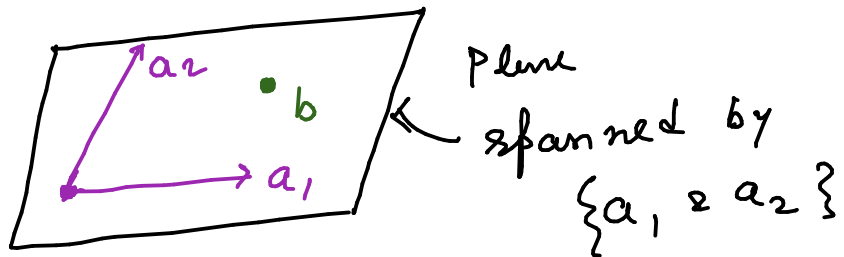
$$x_1 + x_2 + 2x_3 = 3 \quad \Rightarrow x_1 = 1$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} x_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$a_1 \quad a_2 \quad a_3 \quad b$

$\{a_1, a_2, a_3\}$ are not linearly independent.

Can 'b' be
obtained
using
 a_1 & a_2
only?



↓
Yes $\Rightarrow$ Solution exists.

↓
'b' can be obtained by
taking l.c. of a_1 & a_2

• Case - III

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 5 \end{array} \right]$$

$A \quad b$

Called "inconsistent"
↓
system of
eq?

No solution

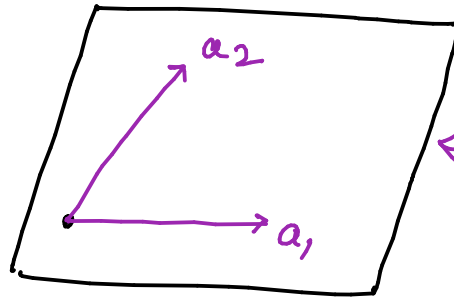
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Logically inconsistent

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} x_3 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$a_1 \quad a_2 \quad a_3 \quad b$

• $b \leftarrow$ Not on the plane



$\leftarrow$ Plane in 3-D space.
 $\leftarrow$ Spanned by a_1 & a_3

b can not be obtained
 by taking L.C. of columns of A

$\rightarrow$ Gaussian Elimination Process
 may break when '0'
 appears in pivot
position.

$\downarrow$
 This may be avoided sometimes
 by doing row interchange.

$\downarrow$
 the $(i, i)^{th}$
 position

$\downarrow$
 Sometimes it may NOT at all
 possible!

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$$\begin{bmatrix} \textcircled{1} & 1 & 2 & : & 3 \\ 1 & 1 & 1 & : & 1 \\ 2 & 2 & 3 & : & 4 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 1 & 2 & : & 3 \\ 0 & \textcircled{0} & 1 & : & 2 \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

→

$$\begin{bmatrix} \textcircled{1} & 1 & 2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

↓

$$\begin{bmatrix} \textcircled{1} & 1 & 2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑

Gaussian Elimination
Process has stopped!
No non-zero element
in (2,2) posⁿ.

Modified Gaussian
Elimination

Row Echelon Form

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

↓

Pivot elements

1, -2, 3

Pivot positions:

(1,1) (2,3) & (3,5)

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↑

Row Echelon Form

↓

Look at the columns of

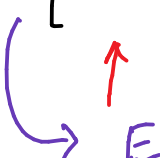
A at the positions

(1), (3), (5)

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

Basic Columns of A

$$\left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 1 & 3 & 3 & x_1 \\ 0 & 0 & \textcircled{-2} & -2 & -2 & x_2 \\ 0 & 0 & 0 & 0 & \textcircled{3} & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 & x_5 \end{array} \right] -$$



The variables corresponding to the Basic columns are called "Basic Variables".


 $x_1, x_3 \text{ \& } x_5$

The other variables are called

"Free variables" $\longrightarrow x_2 \text{ \& } x_4$


 in which column
 of E the pivot
 elements do not
 appear.

→ Solving

$$b = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \rightarrow b = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 = 3$$

$$x_1 + x_3 = 1$$

$$2x_1 + x_2 + 3x_3 = 4$$

→ Non-homogeneous
system of eq's

$$\downarrow [A : b]$$

$$\begin{bmatrix} \textcircled{1} & 1 & 2 & : & 3 \\ 1 & 0 & 1 & : & 1 \\ 2 & 1 & 3 & : & 4 \end{bmatrix}$$

$$\text{Put } \boxed{b = 0}$$

Homogeneous eq's.

↓

$$[E : c] \downarrow$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

→

$$\begin{bmatrix} \textcircled{1} & 1 & 2 & : & 3 \\ 0 & \textcircled{1} & 1 & : & 2 \\ 0 & 0 & \underline{0} & : & \underline{0} \end{bmatrix}$$

↳ solⁿ possible.

Look at the set of corresponding

eq's for $b = 0$

⇓

$$c = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

↓

$$x_2 = -x_3$$

$$x_1 = -x_2 - 2x_3$$

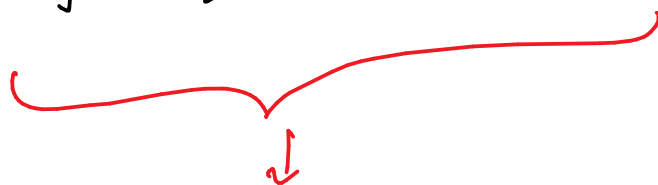
$$= x_3 - 2x_3 = -x_3$$

x_3 is a free variable

line pivot position
is not in 3rd col.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

x_3 is free.



$Ax = 0 \Rightarrow$ The solⁿ set is
in Null space of A
 $N(A)$

• We now solve for non-homogeneous a_2 .

$$[E : C]$$

$\downarrow$

$$x_1 + x_2 + 2x_3 = 3$$

$$x_2 + x_3 = 2$$

$$x_2 = 2 - x_3$$

$$x_1 = 3 - x_2 - 2x_3$$

$$= 3 - (2 - x_3) - 2x_3$$

$$= 1 - x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{\uparrow} x_3$$

- The solⁿ consisting of 2 parts

$$x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x_n = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

General solⁿ:

$$x = x_p + x_n$$

↑
Particular
solⁿ

↑
Homogeneous
solⁿ?

$$Ax = b$$

↓

$$A(x_p + x_n) = b$$

↑↑

$$Ax_n = 0$$

Basic columns

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 3 & 6 & 1 & 4 & 7 \end{array} \right]$$

$\underbrace{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} 3 \end{bmatrix}}_{b} \quad \underbrace{\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}}_{b}$

Range space of A

$$R(A) = \text{span} \{g_1, g_2\}$$

"

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccccc} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 5 & 5 & 5 \end{array} \right]$$

↓

$$\left[\begin{array}{ccccc} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

→

$$\begin{array}{c} [E : c] \\ \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 2 & 3 & 4 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Look at Homogeneous part first

$$\text{Set } b = 0 \Rightarrow c = 0$$

$$x_1 + 2x_2 + 2x_3 + 3x_4 = 0$$

$$x_3 + x_4 = 0$$

$$\Rightarrow x_3 = -x_4$$

$$x_1 = -2x_2 - 2x_3 - 3x_4$$

$$= -2x_2 - x_4$$

Free variables

$$x_2 \text{ \& } x_4$$

Basic variables

$$x_1 \text{ \& } x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4$$


 x_n

• Consider the Non-homogeneous Part

$$x_1 + 2x_2 + 2x_3 + 3x_4 = 4$$

$$x_3 + x_4 = 1$$

$$\Rightarrow x_3 = 1 - x_4$$

$$x_1 = 2 - 2x_2 - x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{h_1} x_2 + \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{h_2} x_4$$

$$x = x_p + x_n$$

Since

$$Ax_n = 0$$

$\Downarrow$

x_n is lin. combinatⁿ of

$$h_1, h_2$$

$\Downarrow$

the null space basis of A is:

A is:

$$N(A) = \text{span}\{h_1, h_2\}$$

$\downarrow$

$$N(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$[A; b] \rightarrow [E; c]$$

$\downarrow$

$$\begin{bmatrix} \textcircled{1} & 2 & 2 & 3 & 4 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{pivot} \\ \leftarrow \text{pos}^n \text{ rows} \end{matrix} \begin{matrix} \text{will form} \\ \text{basis for } R(A^T) \end{matrix}$$

$$R(A^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

- Rank of A = Number of pivot elements

= Number of non-zero rows of E

$$[A:b] \rightarrow [E:c]$$

Row Echelon form

= Number of basic columns of A

- The nonzero rows of E span $\mathcal{R}(A^T)$
- The basic columns of A span $\mathcal{R}(A)$

- Let $x_n = h_1 x_{f_1} + h_2 x_{f_2} + \dots + x_{f_{n-r}} h_{n-r}$

Homogeneous solⁿ

$$r = \text{rank}(A)$$

↓

- The null space is span $\{h_1, h_2, \dots, h_{n-r}\}$

→ To compute basis for $\mathcal{N}(A^T)$

↓

Start $[A:I]$

Identity matrix of size same as rows of A

$$[A : I]$$

↓

$$[E : N]$$

↓

N is obtained from I
by performing same
operation on I as
done in A to
reach E .

Look at the zero rows of E

The corresponding rows of N are the
basis for A^T .

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array} \right]$$

↓

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 5 & 5 & -3 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array} \right]$$

E

N

$$\mathcal{R}(A^T) = \text{span} \left\{ \begin{bmatrix} 1/3 \\ -5/3 \\ 1 \end{bmatrix} \right\}$$

→ Solving using LU & QR Decomposition:

$$A \longrightarrow LU$$

L : Lower triangular matrix

U : Upper triangular matrix

$$Ax = b$$

$$\Rightarrow LUx = b$$

Let $Ux = c \dots \textcircled{1}$

$\Rightarrow Lc = b \dots \textcircled{2}$

↑
given

↑
given

First solve $Lc = b$ & obtain 'c'



Forward substitution

Then

solve

$$Ux = c$$

↑
given

↑
computed



Solve for x by using

back-substitution

$$A = QR$$

↑

Q is orthogonal

R is upper triangular

$$Ax = b$$

$$\Rightarrow QRx = b$$

$$\Rightarrow Rx = \underbrace{Q^T b}_{= c}$$

$$\Rightarrow Rx = c$$

Solve for x using back substitution.