

NL/15/10/25

$$\min f_0(x)$$

$x \in S \leftarrow$ gives a characterization
of constraints.

$$f_0: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$f(x) = x_1 + 3x_2 \leftarrow \mathbb{R}^2 \rightarrow \mathbb{R} \quad /$$

Characterization
of the
constraint
set S

$$f_i(x) \leq 0 \quad i = 1, 2 \dots k$$

$$h_j(x) = 0 \quad j = 1, 2 \dots m$$

$$f_i: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$h_i: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\min_{x \in \mathbb{R}^n} f_0(x)$$

$$\text{s.t.} \quad f_i(x) \leq 0$$

$$h_j(x) = 0$$

- The properties of x
- The nature of f_i, h_i
- The properties/nature of S .

$$x(k+1) = 2x(k)$$

$$k = 0, 1, 2, \dots$$

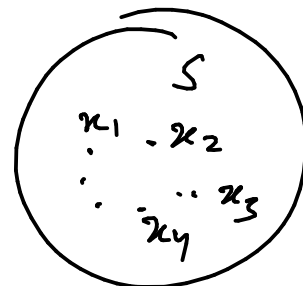
$$x(0)$$

$$x(1) = 2x(0)$$

→ Sequence :

$$\{x_k\} \quad k = 0, 1, 2, \dots$$

• Convergence of a sequence



A seq. $\{x_k\}$ is convergent

to a point x $\{x_k\} \rightarrow x$

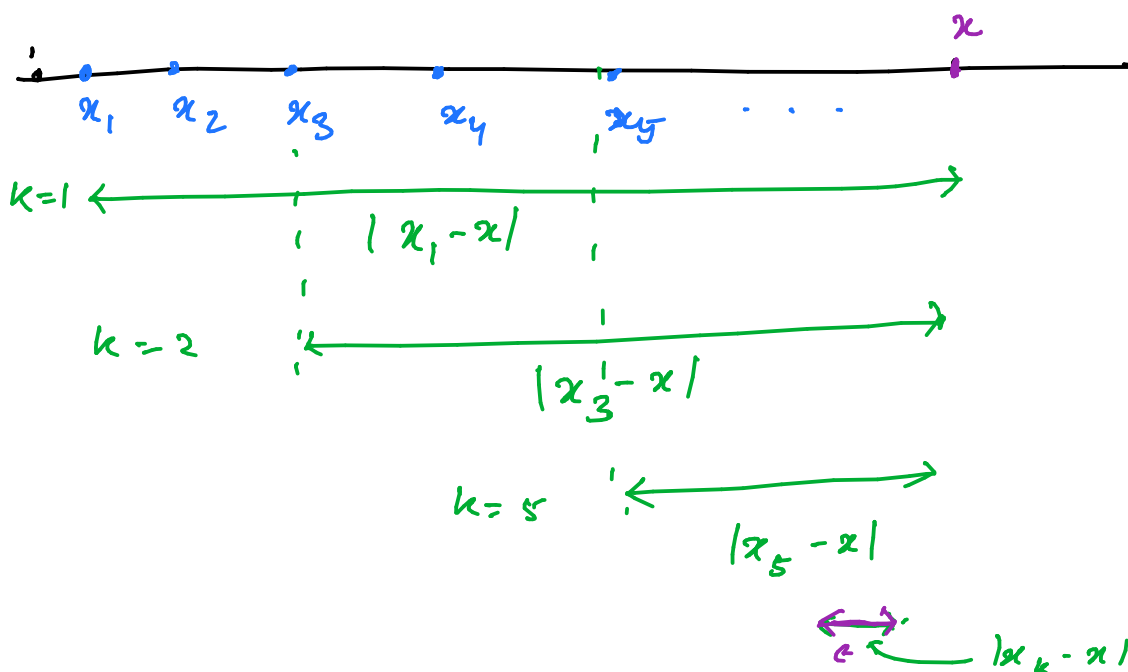
if

$$\text{or } \lim_{k \rightarrow \infty} x_k = x$$

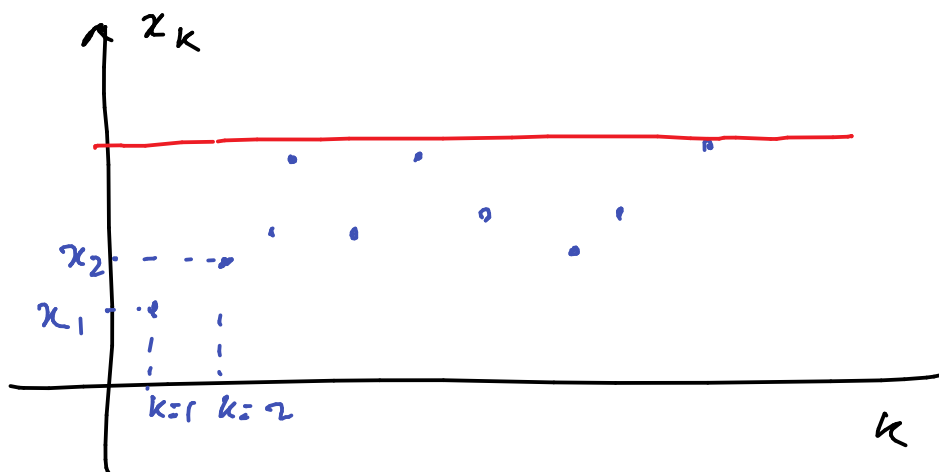
for every given ϵ

there exists some N s.t.

$$|x_k - x| < \epsilon \quad \text{for every } k \geq N$$



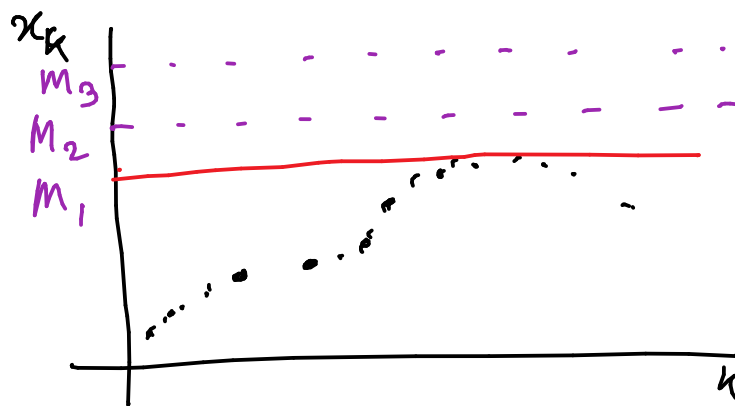
→ Bounded Sequence



$$|x_k| \leq M \quad \text{for all } k$$

• Supremum of a set :

$$\sup(S)$$



$$S \leftarrow \{x_k\}$$

$$\begin{aligned} x &\leq M_1 \\ x &\leq M_2 \\ x &\leq M_3 \end{aligned}$$

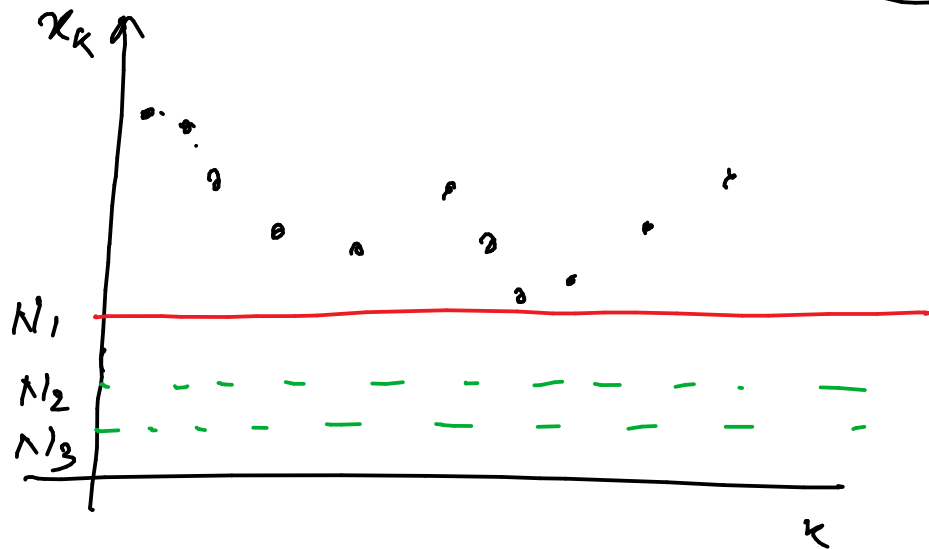
$$|x| \leq \gamma_{\sup} \quad \forall x \in S \quad M_1 \text{ is qualified to be } \gamma_{\sup}$$

$\gamma_{\sup}$ is the lowest upper bound of $x \in S$

$\gamma_{\sup}$ is the supremum of S .

Infimum of a set

$x_k \in S$



$$\begin{aligned} N_1 &\leq |x| \\ N_2 &\leq |x| \\ N_3 &\leq |x| \end{aligned}$$

$$r_{\inf} \leq |x|$$

$$|x| \leq r_{\inf} \quad \forall x \in S$$

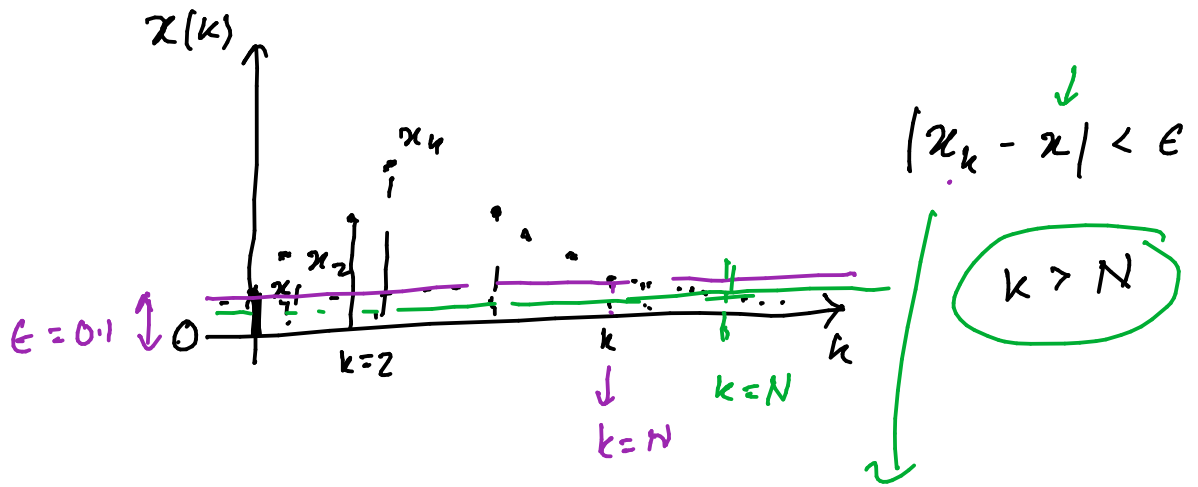
when $r_{\inf}$ is the largest
lower bound of $|x|$.

$$r_{\inf} = N_1$$

NL/17/10/25

$$x(k+1) = \alpha x(k)$$

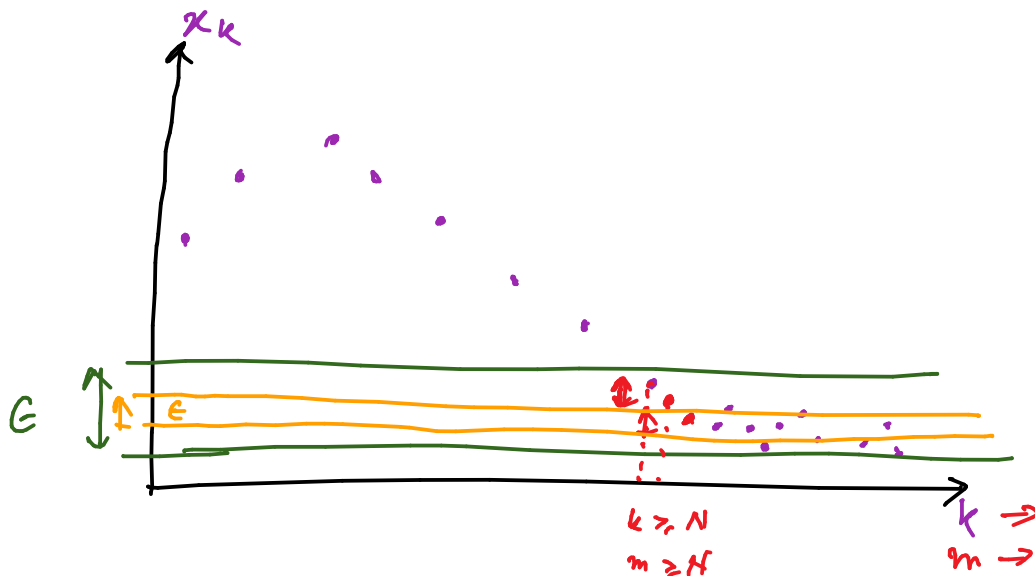
$$k = 0, 1, \dots$$



→ Cauchy Sequence

A sequence $\{x_k\}$ is said to be Cauchy sequence if for every $\epsilon > 0$, $\exists$ some N s.t.

$$|x_k - x_m| < \epsilon \quad \forall \quad k \geq N \quad \& \quad m \geq N$$



- Every Cauchy sequence is bounded.
- Every real valued Cauchy sequence is convergent.

$$\rightarrow \{x_k\} \rightarrow x_k \in \mathbb{R}^n$$

$$x_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$\{x_k\} = \begin{bmatrix} \{x_{1k}\} \\ \{x_{2k}\} \\ \vdots \\ \{x_{nk}\} \end{bmatrix} \rightarrow \text{is a sequence of real numbers}$$

$$\{x_k\} \rightarrow \{x\} \quad x \in \mathbb{R}^n$$

$$\begin{bmatrix} \{x_{1k}\} \\ \{x_{2k}\} \\ \vdots \\ \{x_{nk}\} \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$|x_k - x| < \epsilon$$

for all $k \geq N$



$$\|x_k - x\| < \epsilon$$

for all $k \geq N$

Convergence of a sequence in $\mathbb{R}^n$

Open Ball in $\mathbb{R}^n$

$$B(\bar{x}, \epsilon) := \left\{ x \mid \|x - \bar{x}\|_2 < \epsilon \right\}$$

open Ball ϵ is called radius
 $\bar{x}$ is center

$\mathbb{R}^2$
 $\mathbb{R}^3$

- A sequence is convergent if for every $\epsilon > 0$, one can find N s.t.
$$x_k \in B(\bar{x}, \epsilon) \quad \forall k \geq N$$

Limit point of

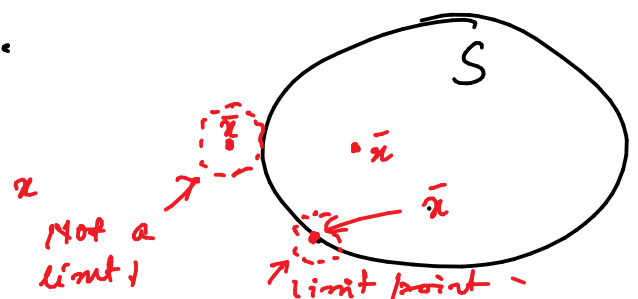
→ A sequence $\{x_k\}$ is bounded

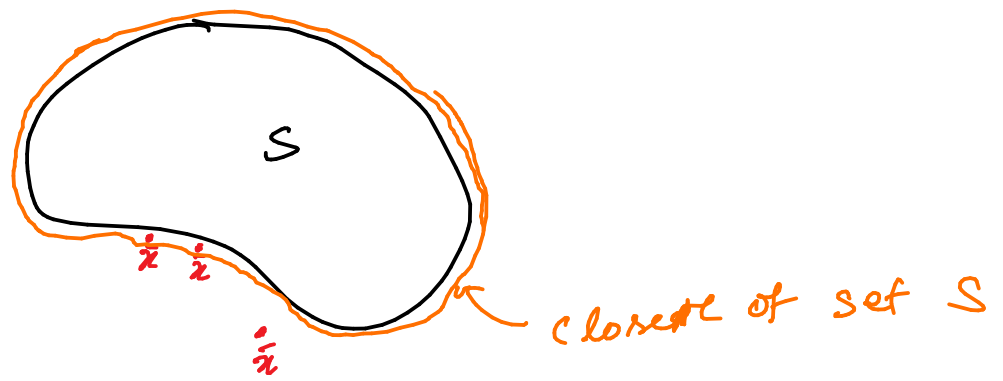
$$\|x_k\| < r$$

r is any positive no.

$\forall$ all k

→ A point $\bar{x}$ is a limit point of a set S if every open ball around it contains at least one point from S , except $\bar{x}$.





$(1, 2)$ ✓

↓
closure points of $(1, 2) \rightarrow 1, 2$

→ closure of a set

$$\hookrightarrow Cl(S) = S \cup \text{all its limit points}$$

Set $S = (1, 2)$

its closure points 1, 2

$$Cl(S) = S \cup \{1, 2\}$$

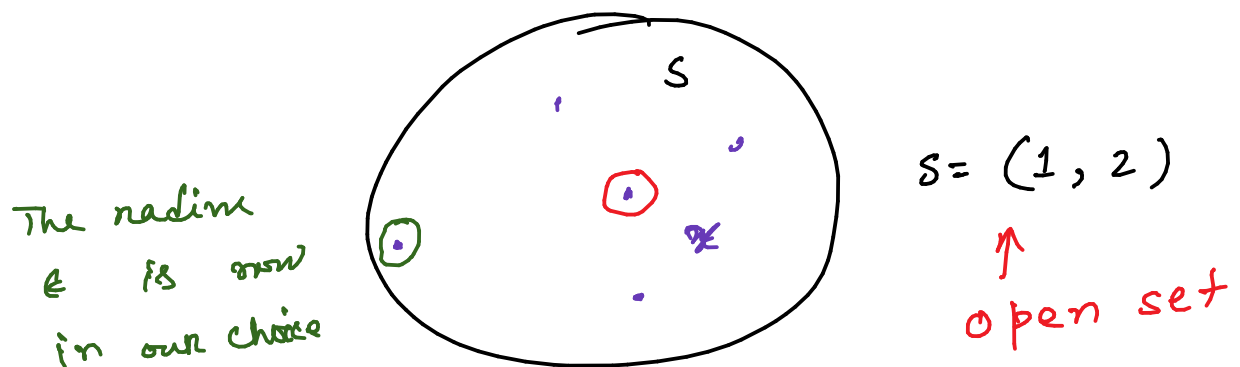
$$= [1, 2]$$

A set S is closed if it is
equal to its closure.

$$S = [1, 2] \leftarrow \text{closed set}$$

• $\rightarrow$ Open set S : the complement of a set S is closed.
 $\downarrow$

A set S is open if for every $x \in S$, there is an open ball $\mathcal{O}(x, \epsilon)$ is entirely contained in X .



Open ball $\mathcal{O}(\bar{x}, \epsilon) = \{x \mid \|x - \bar{x}\| < \epsilon\}$

Closed ball $\mathcal{D}(\bar{x}, \epsilon) = \{x \mid \|x - \bar{x}\| \leq \epsilon\}$

$\rightarrow$ Bounded Set

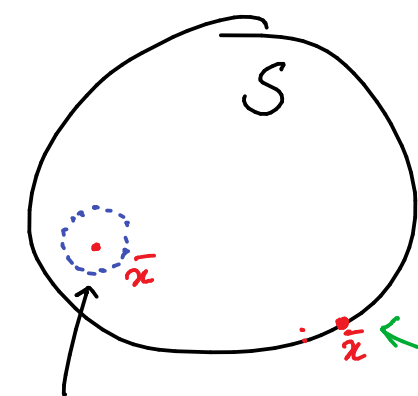
A set S is bounded if for every $x \in S$ we have:

$$\|x\| \leq r$$

$\rightarrow$ A set is said to be "Compact" if it is closed & bounded.

NL/21/10/25

- Closed set
 - Open set
 - Bounded set
 - Compact set (closed & bounded)
-
- Interior Point of a Set:



• Boundary point.

$\bar{x}$ is an interior point

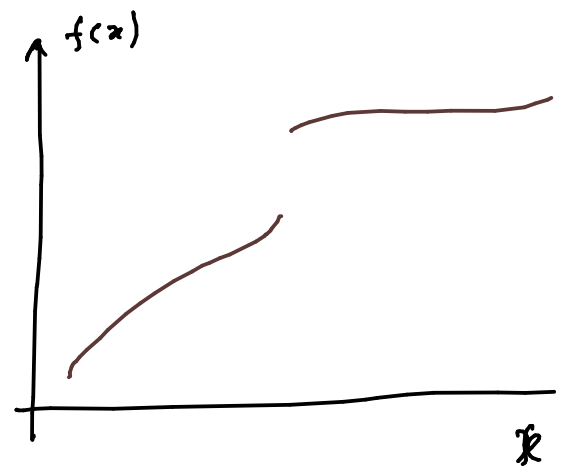
boundary point, which is not an interior point.

→ Continuous function:

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f(x) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix}$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

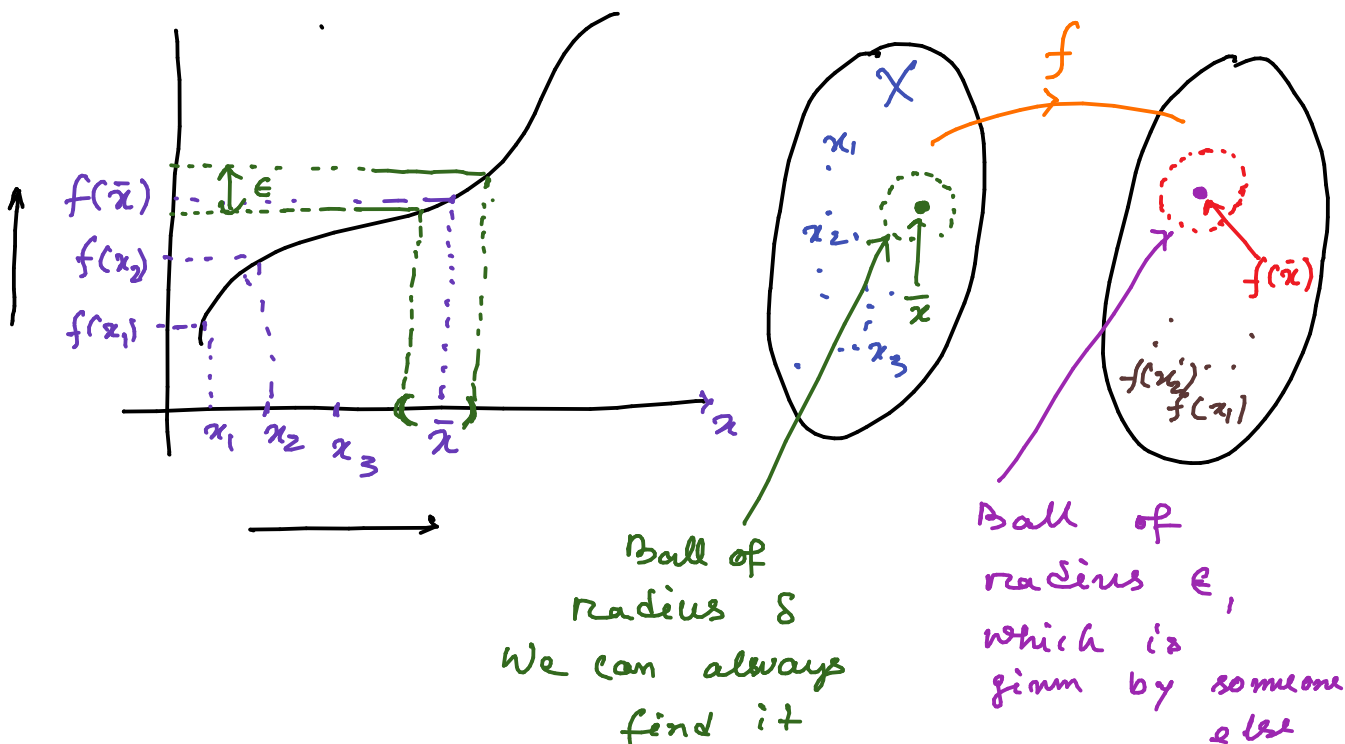


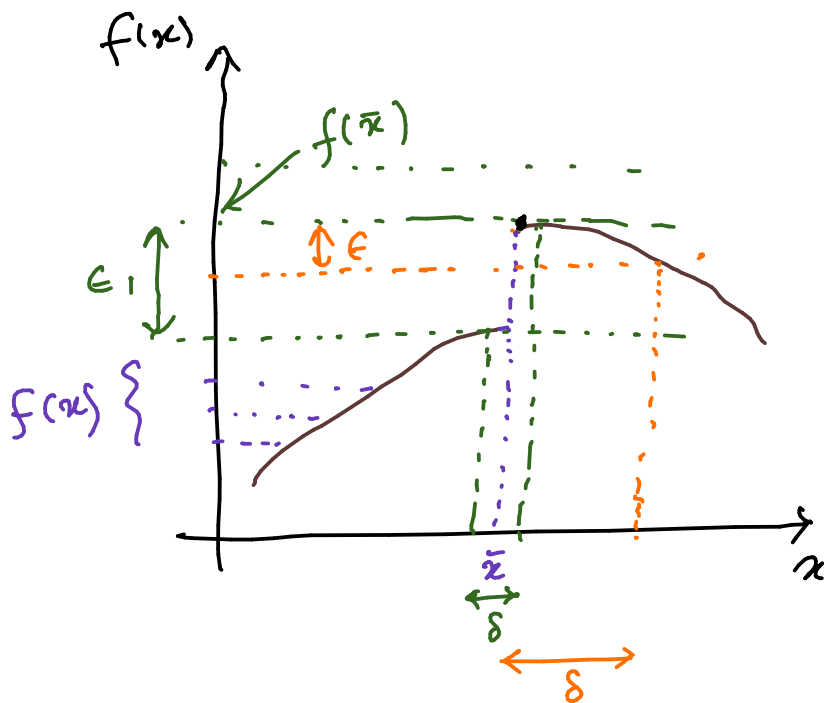
- Left side limit
- Right side limit

- A function f is said to be continuous at point $\bar{x} \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$, s.t.

$$\|f(x) - f(\bar{x})\|_2 < \epsilon \quad \text{whenever} \quad \|x - \bar{x}\|_2 < \delta$$

$\forall x \in X$





→ The Definition in terms of sequence

A funⁿ $f : X \rightarrow \mathbb{R}^m$ where $X \subseteq \mathbb{R}^n$ is continuous at point $\bar{x} \in X$ if for every sequence $\{x_k\}$ of points of X converging to $\bar{x}$, the sequence $\{f(x_k)\}$ must converge to $f(\bar{x})$.

$$\{x_k\} \longrightarrow \bar{x}$$

$$\{f(x_k)\} \longrightarrow f(\bar{x})$$

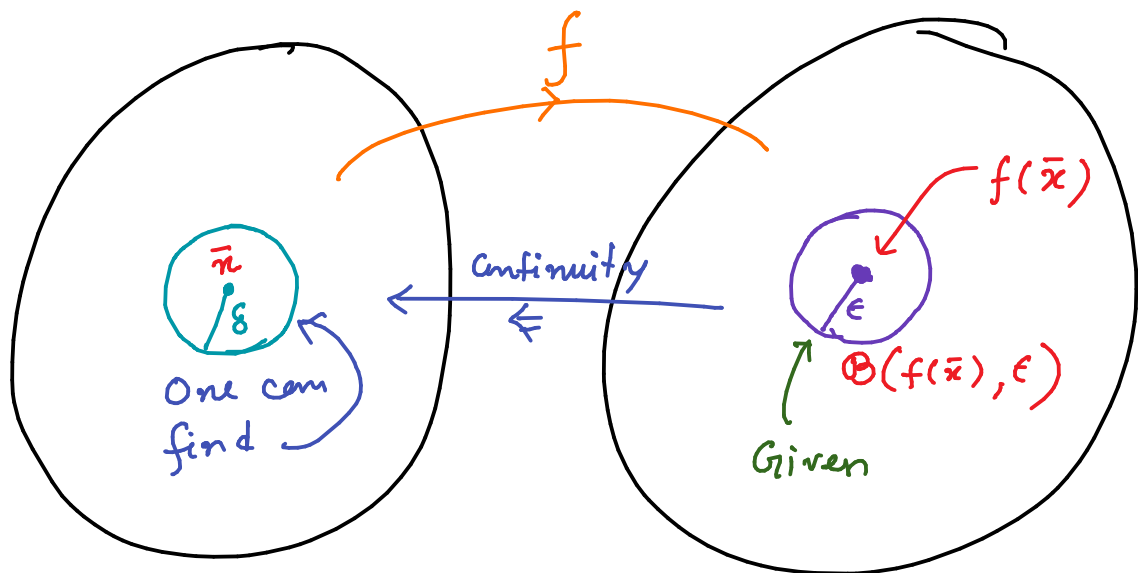
$$\mathbb{R}^n \rightarrow \mathbb{R}$$

$$x_k \geq \bar{x}$$

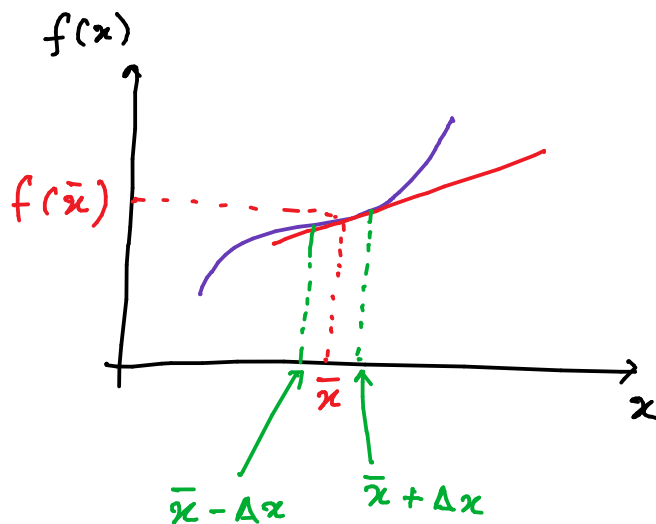
$$\bar{x} \leq x_k$$

functional

NL/24/10/25



→



Scalar real
valued
funt'

$$f: X \rightarrow \mathbb{R}$$

$$X \subseteq \mathbb{R}$$

$\bar{x} \in I$
↑
open interval

slope :

$$\frac{f(\bar{x} + \Delta x) - f(\bar{x})}{(\bar{x} + \Delta x) - \bar{x}}$$

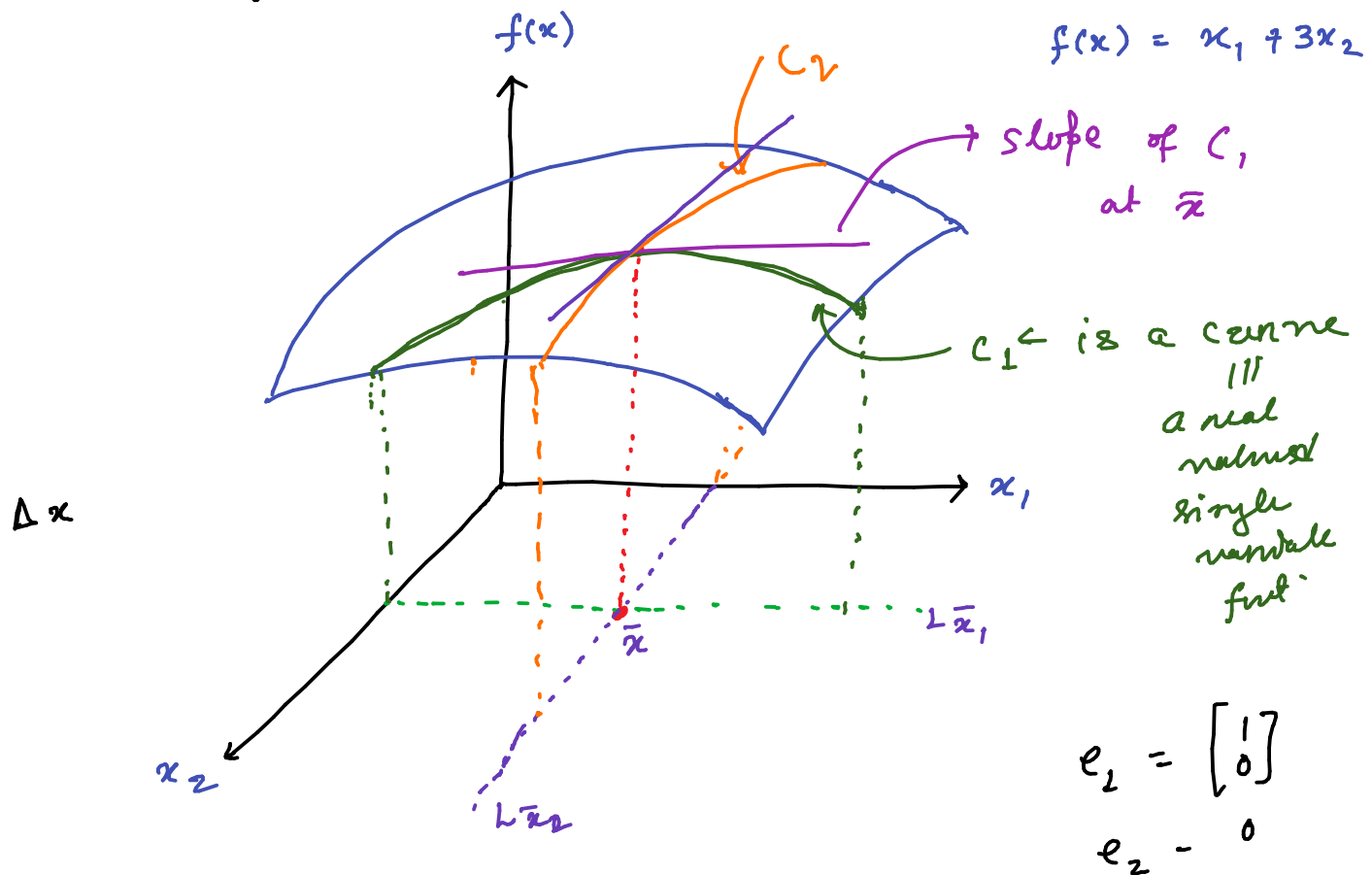
← change in $f(x)$ -axis

← change in x -axis

$$\left. \frac{df}{dx} \right|_{\bar{x}} = \lim_{\Delta x \rightarrow 0} \frac{f(\bar{x} + \Delta x) - f(\bar{x})}{\Delta x}$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$f : X \rightarrow \mathbb{R} \quad \text{when} \quad X \subseteq \mathbb{R}^2$$



How I represent
 Δx here?
 $\Delta x = \theta e_1$

$$\frac{f(\bar{x} + \theta e_1) - f(\bar{x})}{\theta} \leftarrow \text{slope of tangent line of } C_1 \text{ at } \bar{x}$$

$$\left. \frac{\partial f}{\partial x_1} \right|_{\bar{x}} = \lim_{\theta \rightarrow 0} \frac{f(\bar{x} + \theta e_1) - f(\bar{x})}{\theta}$$

$$\frac{\partial f}{\partial x_1} = \lim_{\theta \rightarrow 0} \frac{f(x + \theta e_1) - f(\bar{x})}{\theta}$$

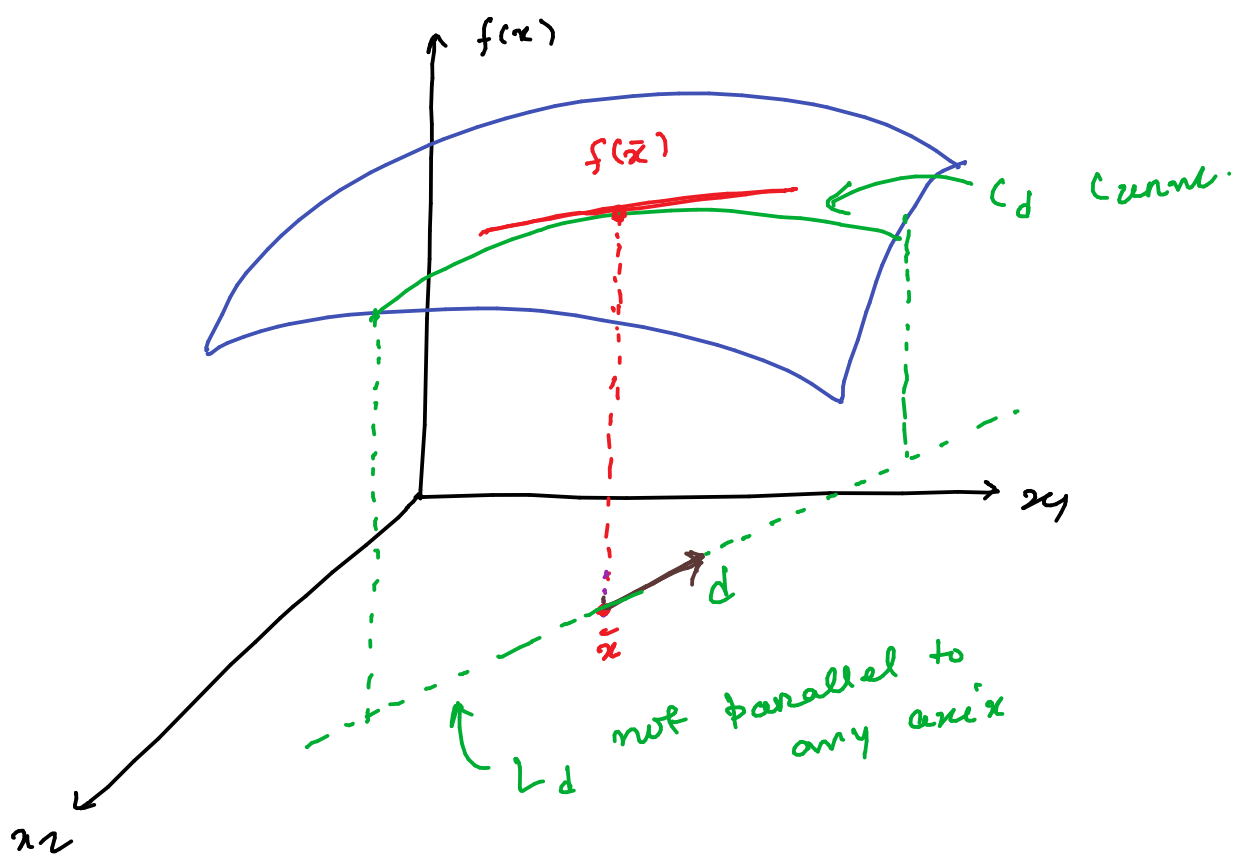
$$\frac{\partial f}{\partial x_2} = \lim_{\theta \rightarrow 0} \frac{f(x + \theta e_2) - f(\bar{x})}{\theta}$$

$$f : X \longrightarrow \mathbb{R} \quad X \subseteq \mathbb{R}^n$$

i^{th} partial derivative

$$\frac{\partial f}{\partial x_i} = \lim_{\theta \rightarrow 0} \frac{f(x + \theta e_i) - f(x)}{\theta}$$

$$\nabla f(x) \Big|_{\bar{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \Big|_{\bar{x}} \leftarrow \begin{array}{l} \text{Gradient vector} \\ \text{of } f(x) \end{array}$$



How do I represent Δx here?

Any point on the line can be represented as $\bar{x} + \theta d$

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

when d is direct? θ is a parameter

$$f'(\bar{x}, d) = \lim_{\theta \rightarrow 0} \frac{f(\bar{x} + \theta d) - f(\bar{x})}{\theta}$$

Directional Derivative

$$f(x) = f(g(\theta))$$

↑

We are looking the points x which are lying on L_d line.

Representing C_d curve $x = \bar{x} + \theta d$ is a composite funtⁿ in θ

↓

Chain Rule to evaluate deriv

$$x = \bar{x} + \theta d = \begin{bmatrix} \bar{x}_1 + \theta d_1 \\ \bar{x}_2 + \theta d_2 \end{bmatrix} \begin{matrix} \leftarrow x_1 \\ \leftarrow x_2 \end{matrix}$$

$$\frac{df}{d\theta} = \frac{\partial f}{\partial x_1} \frac{dx_1}{d\theta} + \frac{\partial f}{\partial x_2} \frac{dx_2}{d\theta}$$

↑

$$f'(\bar{x}, d) = \frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2$$

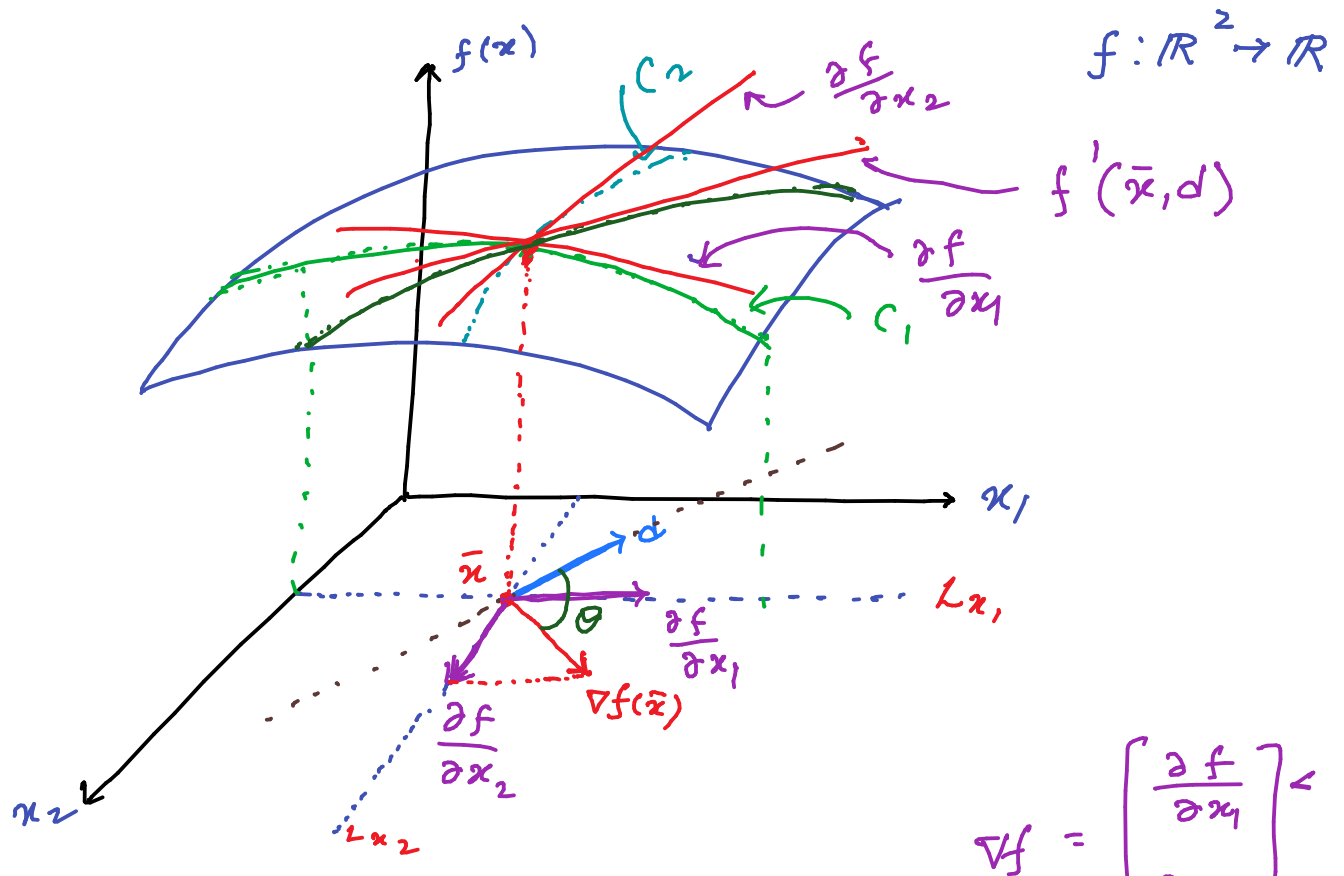
$$= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= (\nabla f(x))^T d$$

- The Relatⁿ betⁿ Directional derivatⁿ & gradient funtⁿ:

$$f'(\bar{x}, d) = \nabla f(x)^T d$$

NL/28/10/25



$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$\frac{\partial f}{\partial x_i} \leftarrow i^{\text{th}} \text{ partial derivative}$

$f'(x, d) \leftarrow \text{Directional derivative}$

$$f'(x, d) = \nabla f(x)^T d$$

$$\langle \nabla f(x), d \rangle = \nabla f(x)^T d = \|\nabla f(x)\| \|d\| \cos \theta$$

Assume that $\|d\| = 1$

$\Downarrow$

$$\nabla f(x)^T d = \|\nabla f(x)\| \cos \theta$$

$$-1 \leq \cos \theta \leq 1$$

$\nabla f(\bar{x})^T d$ is maximum when $\theta = 0^\circ$
 $\Downarrow$

$\left\{ \begin{array}{l} d \text{ \& } \nabla f(\bar{x}) \text{ are aligned in} \\ \text{same direction.} \end{array} \right.$
 Steepest ascent direction

$\nabla f(\bar{x})^T d$ is minimum when $\theta = 180^\circ$
 $\Downarrow$

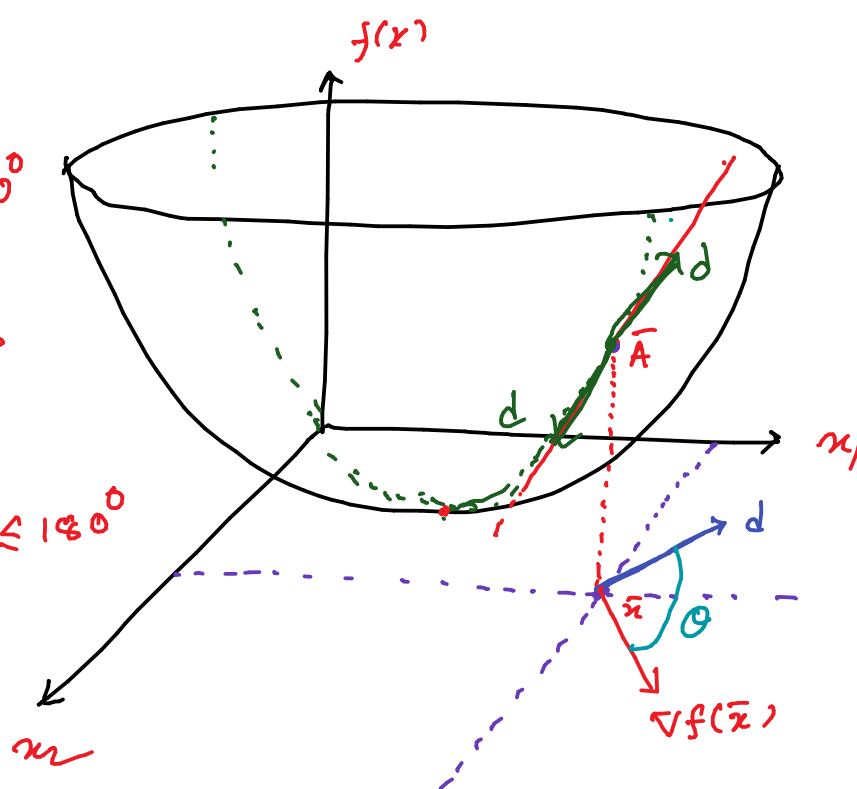
$\left\{ \begin{array}{l} \nabla f(\bar{x}) \text{ \& } d \text{ are in opposite direction.} \end{array} \right.$
 Steepest descent direction

$\underbrace{\nabla f(\bar{x})^T d}_{f'(\bar{x}, d)} = 0$ when $\theta = 90^\circ$
 $\Downarrow$

$d \text{ \& } \nabla f(\bar{x})$ are orthogonal.

$f'(\bar{x}, d)$ is
 +ve for
 $0 \leq \theta < 90^\circ$

$f'(\bar{x}, d)$ is
 -ve for
 $90^\circ < \theta \leq 180^\circ$

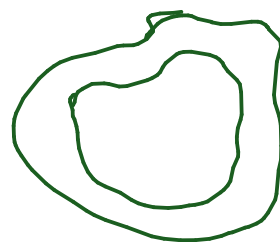
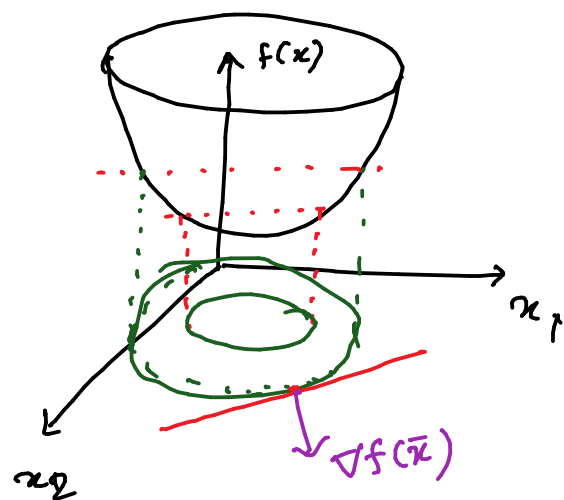


$\bar{A} = f(\bar{x})$

→ Level curve

$$L_r = \{x \mid f(x) = r\}$$

r is a scalar.



Level sets $L_r = \{x \mid f(x) \leq r\}$

- The gradient vector $\nabla f(\bar{x})$ is always orthogonal to the tangent of level curve at $\bar{x}$

↓
 tangent plane

→ For a vector valued funⁿ

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f = \begin{bmatrix} x_1 \\ x_2 + x_3 \\ x_3^2 \end{bmatrix}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

Jacobian Matrix

$$J = [\nabla f(x)]^T$$

$$= \begin{bmatrix} \nabla f_1(x) & \nabla f_2(x) & \dots & \nabla f_m(x) \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

→ For a scalar valued funⁿ,

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

- assume that the partial derivatives

$\frac{\partial f}{\partial x_i}$ for all i are smooth funⁿ,

$$f(x) = x_1 + 3x_2^3 + 2x_3^2 + x_1^2 + x_2x_3^2$$

$$\frac{\partial f}{\partial x_1} = 1 + 2x_1$$

$$\frac{\partial f}{\partial x_1^2} \leftarrow \frac{\partial f}{\partial x_1 \partial x_1} = 2 \qquad \frac{\partial f}{\partial x_1 \partial x_2} = 0$$

Hessian Matrix of $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$\nabla^2 f(x)$ is a symmetric matrix.