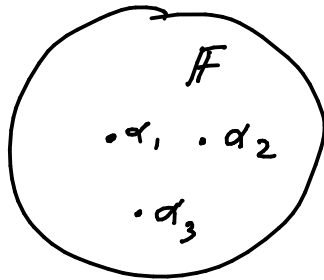


Lecture-3

→ The set of scalars $\mathbb{F}$



→ For every pair α_1, α_2

$$\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1$$

$$\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$$

- $\exists$ one unique element '0' s.t.

$$\alpha_i + 0 = \alpha_i$$

- $\exists$ one unique scalar $-\alpha_i$ s.t.

$$\alpha_i + (-\alpha_i) = 0$$

- $\exists$ unique scalar "1" s.t. $\alpha_i \cdot 1 = \alpha_i$

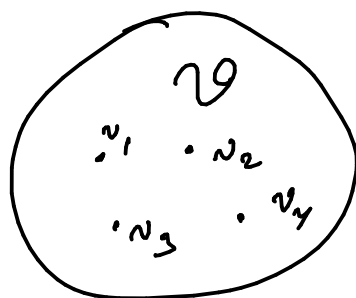
- $\exists$ unique scalar $1/\alpha_i = \alpha_i^{-1}$ s.t. $\alpha_i \cdot \frac{1}{\alpha_i} = 1$

* If the elements of the set $\mathbb{F}$ satisfies the above rules, then we can say that

$\mathbb{F}$ is a field.

Ex : set of Real numbers
complex numbers.

→ A set which satisfies following set of rules:



$$\rightarrow v_1 + v_2 = v_2 + v_1$$

$$\bullet v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$$

- $\exists$ unique element '0' s.t. for $\forall v_i \in V$

$$v_i + 0 = v_i$$

- for every element v_i $\exists$ one unique element $-v_i$ s.t. $v_i + (-v_i) = 0$

- For every $\alpha \in F$ & $v \in V$, the element $\alpha \cdot v \in V$

- For every v , $1 \cdot v = v \in V$

$$\bullet \alpha (v_1 + v_2) = \alpha v_1 + \alpha v_2$$

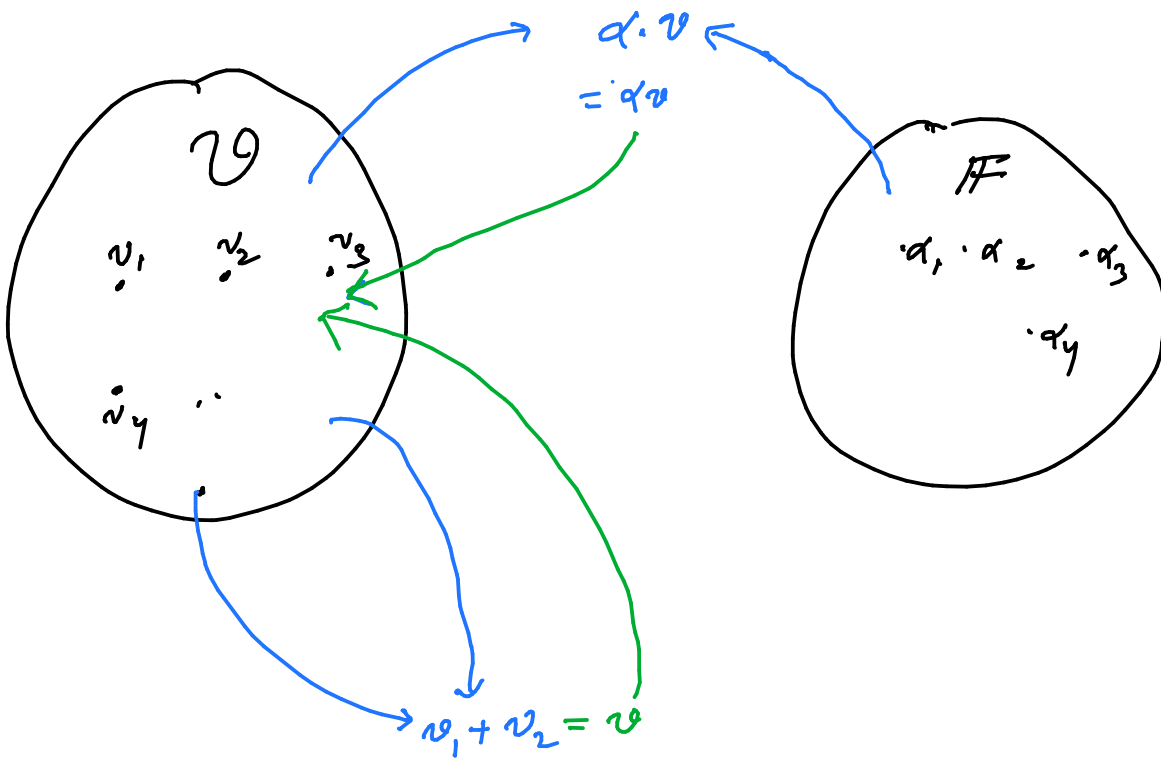
If the elements of set V satisfies all the above rules, then we say that

V is a vector space.



V is a vector space over field F

Scalar multiplication



vector addition

Example of vector space $V \downarrow$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad v_i \in \mathbb{R}$$

$\uparrow$
 $\mathbb{R}^n$

The elements of the set $\mathbb{R}^n$ satisfies the rules for vector space.

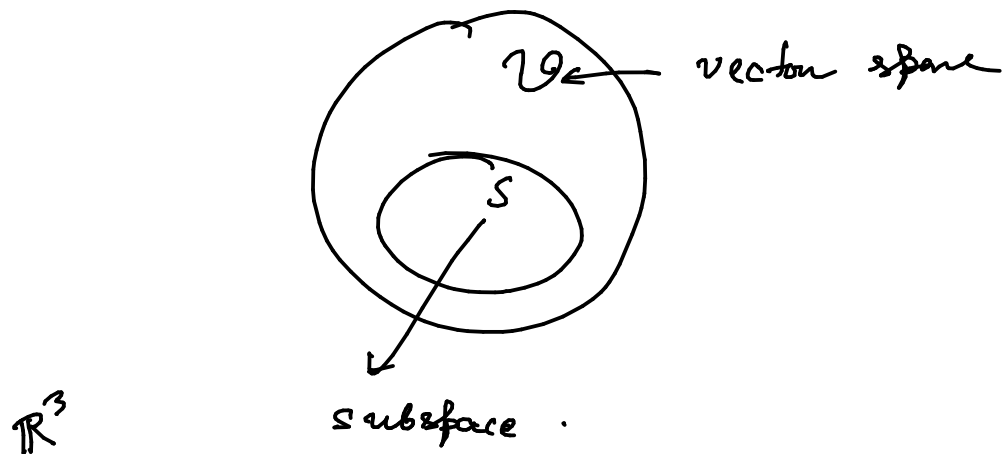
$\Downarrow$

$\mathbb{R}^n$ is a vector space over field $\mathbb{R}$

$\rightarrow$ Euclidean space

→ Subspace :

Let S be a subset of set V . If V is a vector space & the elements of S satisfies the rules of vector space, then S is called subspace.



$$\rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = 2v_1 + (-1)v_2$$

v_3 is linearly dependent on v_1 & v_2

→ In general let say V has elements
 $v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_n v_n = 0 \quad \text{--- } *$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ v_1 & v_2 & \dots & v_n \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$$

$$\underbrace{\begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{bmatrix}}_A \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha} = 0 \quad \text{--- } (*)$$

- If $\alpha \neq 0$ & satisfies $(*)$ then the vectors $v_1, v_2, \dots, v_n$ are linearly dependent.
- If the eqⁿ $(*)$ holds only for $\alpha = 0$, then (trivial solⁿ)
we will say that $v_1, v_2, \dots, v_n$ are not linearly dependent (linearly independent)

$$\checkmark \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

only for $\alpha_i = 0$

→ Primary colours :

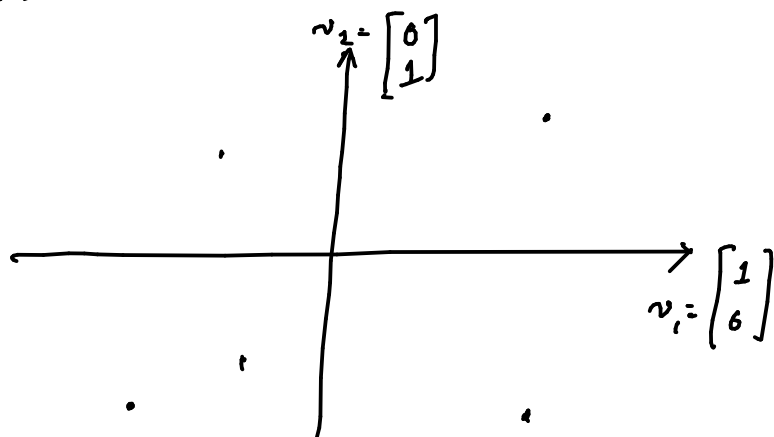


why primary colours

$$\text{red} + \text{green} \rightarrow \text{Yellow}$$

All different colours can be obtained from these above three colours.

→ Like primary colours one can choose a set of vectors in the vector space V s.t. the other elements in V can be generated using those chosen vectors.



For instance in $V = \mathbb{R}^2$, the elements on a plane ($\mathbb{R}^2$) can be generated using the vectors

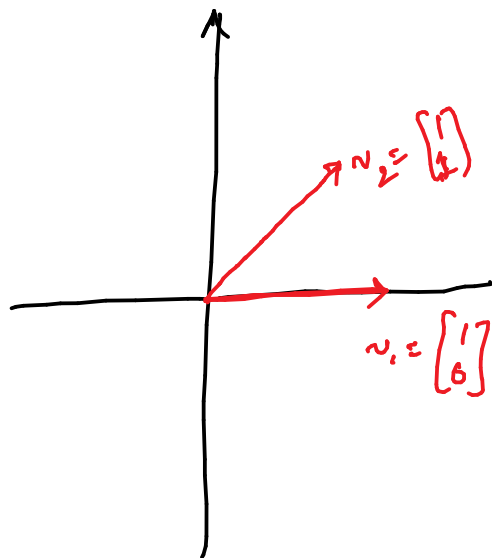
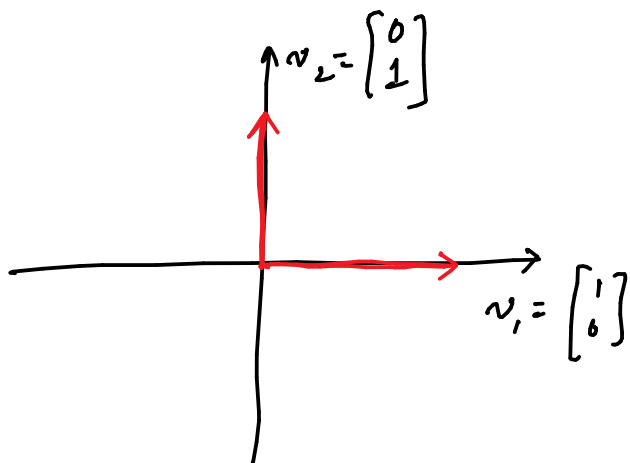
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

→ Like primary colours, we will say that $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as "Basis" for vector space $\mathbb{R}^2$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \dots \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \leftarrow \begin{array}{l} \text{As a basis for } \mathbb{R}^n \\ \text{Standard Basis of } \mathbb{R}^n \end{array}$$

→ Is the choice of basis is unique in a vector space?



$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

→ v_2

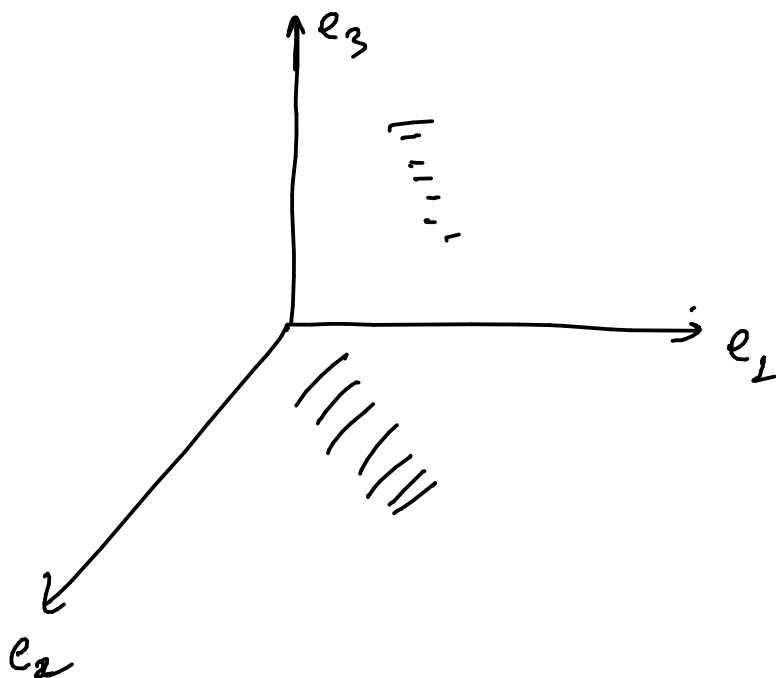
→ The choice of basis for a vector space is not unique.

→ The choice of Basis vectors must satisfy
↓

the vectors are linearly independent

→ To generate the vectors in $\mathbb{R}^3$ we need following three standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



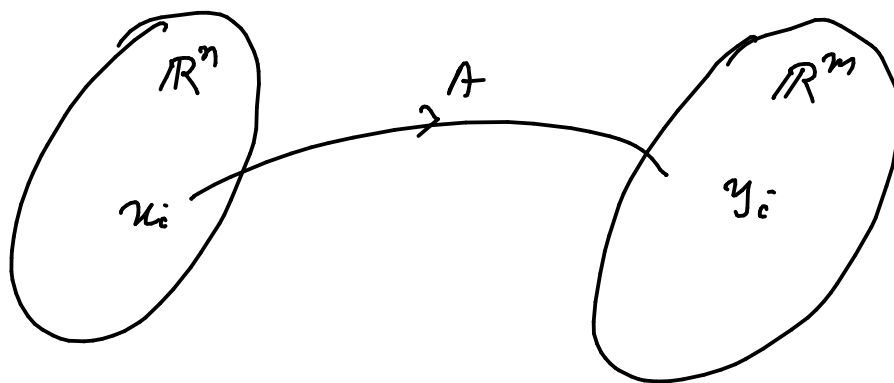
→ One can choose a basis in infinitely many ways for a vector space V . However, the number of vectors in a basis $\mathcal{B}$ will remain same. This number (the no. of elements in $\mathcal{B}$) is called "dimension of vector space" V .

→ Matrix

$$\begin{array}{c}
 \nearrow \\
 \text{Real matrix}
 \end{array}
 \begin{array}{c}
 A \in \mathbb{R}^{m \times n} \\
 \left[\begin{array}{ccc} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right]
 \end{array}
 \quad a_{ij} \in \mathbb{R}$$

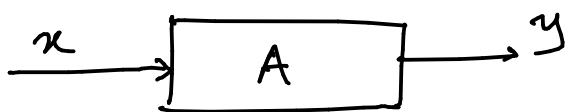
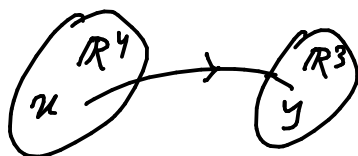
→ If we consider the matrix-vector multiplication.

$A \in \mathbb{R}^{m \times n} \rightarrow$ is a linear operator.



$$Ax_i = y_i$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix}_{3 \times 4} = \begin{bmatrix} y \end{bmatrix}_{4 \times 1}$$



A is a linear operator

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Ax_1 + \alpha_2 Ax_2 = \alpha_1 y_1 + \alpha_2 y_2$$