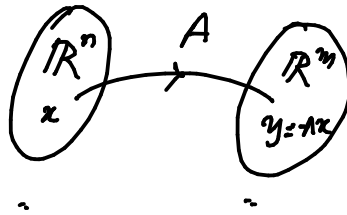


Lecture-4

$$A \in \mathbb{R}^{m \times n}$$



→ Range space on Image space of A

$$\mathcal{R}(A) := \left\{ y \mid y = Ax, x \in \mathbb{R}^n \right\} \dots *$$

$$\mathcal{R}(A) \subseteq \mathbb{R}^m$$

↑
linear combination
of columns of A

↑
Column space of A

• Range space of A^T

$$\mathcal{R}(A^T) = \left\{ x \mid x = A^T y, y \in \mathbb{R}^m \right\} \dots (*)$$

↑
linear combination of rows
of A

$$[y_1, y_2, \dots, y_m] \begin{bmatrix} A \end{bmatrix} \quad \uparrow \quad \text{Row space of } A$$

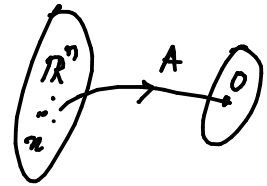
$$\mathcal{R}(A^T) \subseteq \mathbb{R}^n$$

→ Null space or Kernel space of A

$$\mathcal{N}(A) := \{x \mid Ax = 0, x \in \mathbb{R}^n\}$$

Right
null
space
of A

$$\mathcal{N}(A) \subseteq \mathbb{R}^n$$

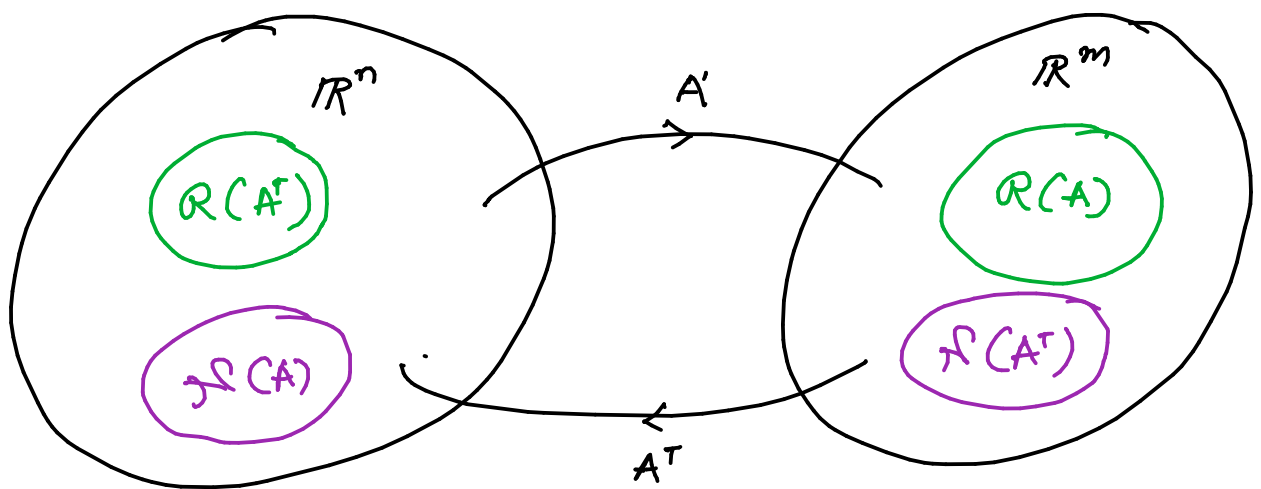


→ Null space of A^T

$$\mathcal{N}(A^T) := \{y \mid A^T y = 0\}$$

$$\mathcal{N}(A^T) \subseteq \mathbb{R}^m$$

Left null space
of A



For $A \in \mathbb{R}^{m \times n}$

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$A^T : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

→ Algorithm to compute

Range space basis & Null space basis

$$\checkmark \begin{bmatrix} \textcircled{1} & 0 & \underline{2} & \underline{1} \\ -1 & 1 & -2 & 1 \\ 0 & 3 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$A: \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

↑ equal to the size of column of A

$$\begin{matrix} c_3 \leftarrow 2c_1 - c_3 \\ c_4 \leftarrow c_1 - c_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & \textcircled{1} & 0 & -2 \\ 0 & 3 & 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{matrix} c_4 \leftarrow 2c_2 + c_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ ↑

basis for $\mathcal{R}(A)$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

↑ ↑

basis for $\mathcal{N}(A)$

$$Ax = 0$$

• Rank of matrix A is equal to the number of pivot element

|||

The dimension of $\mathcal{R}(A)$ is equal to the number of pivot elements = the number of linearly independent columns.

→ Row & Column operations does not change the rank of a matrix.

- Matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$

$\Downarrow$

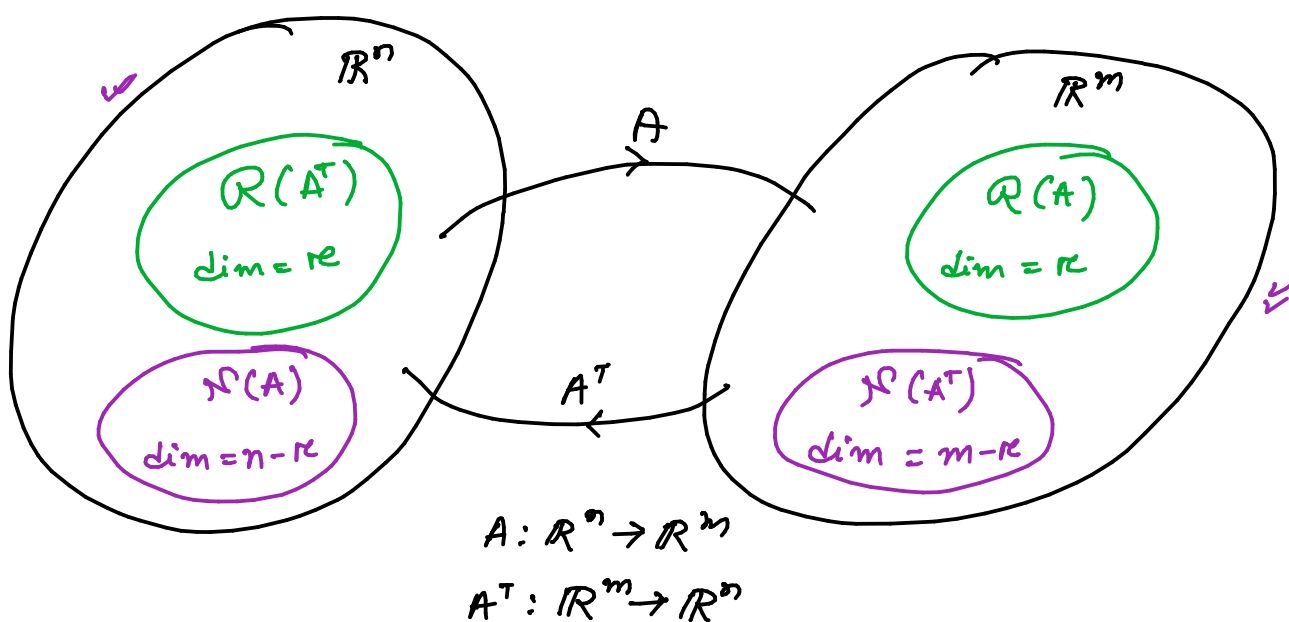
$$\text{rank}(A) = \text{rank}(A^T)$$

there are r linearly independent columns in A .

$\dim[\mathcal{R}(A)] \equiv$ the number of linearly independent columns in $A = r$

$\dim[\mathcal{R}(A^T)] =$ the number of linearly independent columns of $A^T \equiv r$

$$\dim[\mathcal{N}(A)] = n - r \quad \dim[\mathcal{N}(A^T)] = m - r$$

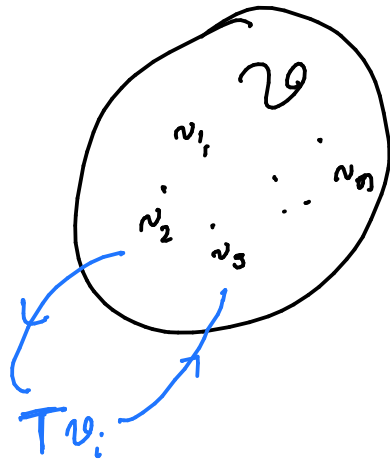


- Rank Plus nullity theorem

$$\dim[\mathcal{R}(A)] + \dim[\mathcal{N}(A)] = n \quad \text{for all } m \times n \text{ matrices}$$

$$\text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^TA)$$

Invariant Space :



The vector space

V is T -invariant
when T is a linear
operator (may be considered
as matrix)

iff

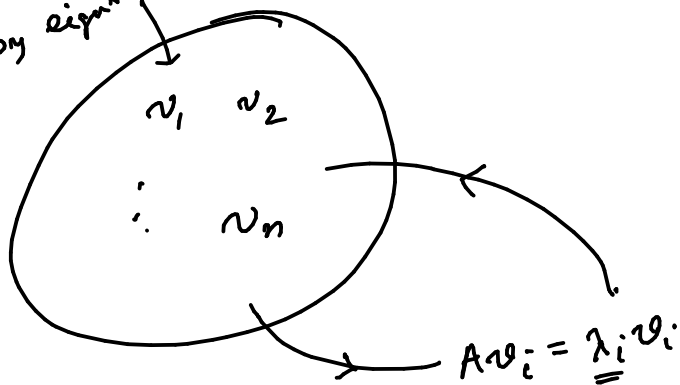
$$Tv_i \in V, \forall v_i \in V$$

$\Downarrow$

V is T -invariant space.

$$A \in \mathbb{R}^{n \times n}$$

V spanned by eigenvectors of A $v_1, v_2, \dots, v_n$ are its eigenvectors associated with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$



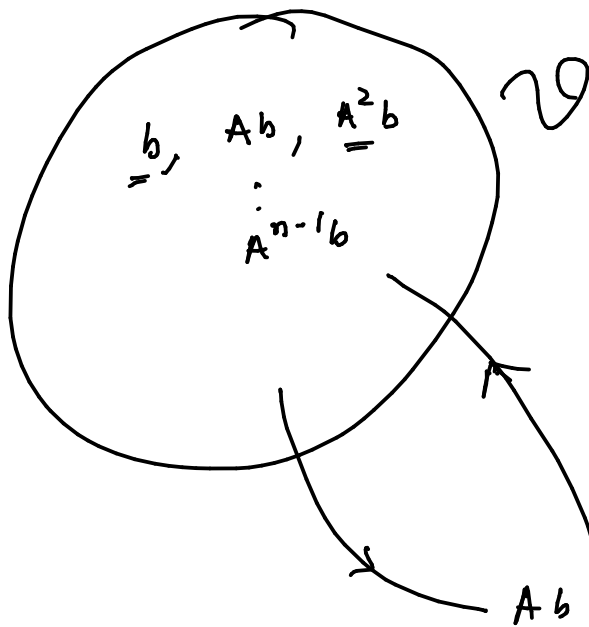
The space spanned by
the eigenvectors of A is
 A -invariant space.

→ Controllability space :

V is constructed by using the following
set of linearly independent vectors:

$$\{ b, Ab, A^2b, \dots, A^{n-1}b \}$$

is also A -invariant.



$$A \cdot (A^{n-1}b) \in U ?$$

$$\underline{\underline{A^n b}}$$

→ Cayley-Hamilton th^m

$$A^n b = () \underline{b} + () \underline{A^2 b} + \dots + () \underline{A^{n-1} b}$$

