Lecture - 10

Internal stability $\equiv$ stability of 4 t.fs of the FI

(the SSRs are stabilizable & detectable)

$$(1 - H_1(s)H_2(s))^{-1}, \quad (1 - H_1(s)H_2(s))^{-1}H_1(s), \quad (1-H_1H_2)^{-1}H_2, \quad (1-H_1H_2)^{-1}H_1H_2$$

$$H_1(s) = \frac{b(s)}{a(s)} \qquad H_2(s) = \frac{y(s)}{x(s)}$$

All of the 4 t.f.s have common denominator

i.e. $\quad a(s)\,x(s) - b(s)\,y(s)$

Some more mathematical tools.

$\mathbb{R}[s] \rightarrow$ set of real coefficient polynomials

$\mathbb{R}(s) \leftarrow$ the set of real rational functions

$$\mathbb{R}(s) := \left\{ g(s) = \frac{n(s)}{d(s)} \; : \; \begin{array}{l} n(s) \in \mathbb{R}[s] \\ d(s) \in \mathbb{R}[s] \end{array} \; \& \; d(s) \neq 0 \right\}$$

$$\delta_\infty(g(s)) = \deg(d(s)) - \deg(n(s))$$

- Number of zeros of $g(s)$ at ∞ = No. of finite poles of $g(s)$ − No. of finite zeros of $g(s)$

$g(s) = \frac{1}{s-1}$ [strictly proper transfer function]

1 zero at ∞

- Number of poles of $g(s)$ at ∞ = No. of finite zeros of $g(s)$ − No. of finite poles of $g(s)$

$g(s) = s - 1$ [Improper t. fs]

↑ one pole at ∞.

$\mathbb{R}[s]$ — polynomials with real coefficients

↓

Irreducible polynomials

$$p(s) = (s^2 - 1)(s^2 + s + 1)$$

- Reducible polynomials $p(s) \in \mathbb{R}[s]$:

A polynomial $p(s) \in \mathbb{R}[s]$ is irreducible over $\mathbb{R}$ if it can not be reducible into the product of two polynomials with coefficients in $\mathbb{R}$.

[$p(s)$ is a non-const. polynomial i.e. $p(s)$ is always dependent on s]

$$p(s) = (s^2 - 1) \in \mathbb{R}[s]$$
$$= \underline{(s-1)}\,\underline{(s+1)}$$

$p(s) = s - 1$ ← irreducible over $\mathbb{R}$

$p(s) = s^2 + s + 1 \quad \leftarrow$ irreducible

$= (s+\lambda_1)(s+\lambda_2)$ when λ_1 & λ_2 are complex

$p(s)$ is reducible over $\mathbb{C}$ $\qquad$ $p(s)$ is irreducible over $\mathbb{R}$

→ Let the polynomial $p(s) \in \mathbb{R}[s]$

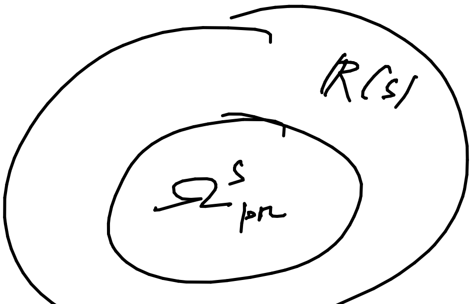
$$p(s) = k\,(s+\lambda_1)^{\nu_1}(s+\lambda_2)^{\nu_2}\cdots(s+\lambda_k)^{\nu_k}(s^2+a_1 s+b_1)^{\nu_1'}$$
$$(s^2+a_2 s+b_2)^{\nu_2'}\cdots(s^2+a_r s+b_r)^{\nu_r'}$$

all of the elements $\left\{ \begin{array}{l} (s+\lambda_i) \\ (s^2+a_i s+b_i) \end{array} \right\}$ are irreducible over $\mathbb{R}$

When two polynomials are <u>co-prime</u>.

- Ω^s_{pr} : the set of proper and stable real rational functions

- $\Omega^s_{pr} := \left\{ g(s) = \dfrac{n(s)}{d(s)} : \begin{array}{l} \deg(d(s)) \geq \deg(n(s)) \\ d(s) \text{ is a stable polynomial} \end{array} \right\}$



For any $\underline{g(s) \in \Omega^{s}_{pr}}$ we can write

$$g(s) = n_u(s) \cdot \frac{n_s(s)}{d_s(s)}$$

$$\frac{s-1}{s+2}$$

$n_u(s) :\to$ unstable polynomial

$n_u(s) = s-1$

$\left.\begin{matrix} n_s(s) \\ d_s(s) \end{matrix}\right\} \to$ stable polynomials

$n_s(s) = 1$
$d_s(s) = s+2$

- **Units in Ω^{s}_{pr}:**

A real rational funtin $u(s) \in \Omega^{s}_{pr}$ is an unit in Ω^{s}_{pr} if there exists $\tilde{u}(s) \in \Omega^{s}_{pr}$ such that $(s) = 1$.

$$u(s) = \frac{1}{\tilde{u}(s)}$$

$$u(s) \in \Omega^{s}_{pr} \to \frac{s+1}{s+2}, \quad \frac{1}{s+2}, \quad \frac{s-2}{s+1}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{s+2}{s+1}, \quad \frac{s+2}{1}, \quad \frac{s+1}{s-2}$$

valid $\tilde{u}(s)$ X $\notin \Omega^{s}_{pr}$

$\to$ The units in Ω^{s}_{pr} are

biproper, stable, minmum phase
(poles & zeros both are in left of $\mathbb{C}$)

Factorization in Ω^s_{pr}

Any real rational function $g(s) \in \Omega^s_{pr}$ can be factored as follows:

$$g(s) = \left[\frac{n_u(s)}{(s+\alpha)^{\nu}}\right]\left[\frac{1}{(s+\alpha)^{q_\infty}}\right] u(s)$$

where

$n_u(s)$ is an unstable polynomial (if there exists)

ν : degree of $n_u(s)$

$\alpha > 0$ (a positive number)

$q = q_\infty + \nu$ where q_∞ is the number of zeros at ∞.

$$u(s) = \frac{n_s(s)(s+\alpha)^q}{d(s)} \in \Omega^s_{pr}$$ is a <u>unit</u>.

q = Degree difference between denominator & numerator of $g(s)$ + degree of unstable polynomial $n_u(s)$

$$g(s) = \frac{n_u(s)}{(s+\alpha)^q} \cdot \frac{n_s(s)(s+\alpha)^q}{d_s(s)}$$

$$= \left[\frac{n_u(s)}{(s+\alpha)^{\nu}}\right] \cdot \left[\frac{1}{(s+\alpha)^{q_\infty}}\right] \cdot \underbrace{\left[\frac{n_s(s)(s+\alpha)^q}{d_s(s)}\right]}_{u(s)}$$

$u(s) \in \Omega_{pr}$ is a unit.

Always $n_u(s)$ can be factored into irreducible factors over $\mathbb{R}$. Let us write it as

$$\boxed{n_u(s) = k\,(s+\lambda_1)^{\nu_1}(s+\lambda_2)^{\nu_2}\cdots(s+\lambda_\mu)^{\nu_\mu}(s^2+a_1s+b_1)^{\nu_1'}(s^2+a_2s+b_2)^{\nu_2'}\cdots(s^2+a_ns+b_n)^{\nu_n'}}$$

when $\lambda_i < 0$ & $k, a_i, b_i \in \mathbb{R}$ — factorization is unique

$$g(s) = \frac{n_u(s)}{(s+\alpha)^{\nu}}\cdot\frac{1}{(s+\alpha)^{q_\infty}}\,u(s) \sim$$

$$= \left[\frac{s+\lambda_1}{s+\alpha}\right]^{\nu_1}\left[\frac{s+\lambda_2}{s+\alpha}\right]^{\nu_2}\cdots\left[\frac{s+\lambda_\mu}{s+\alpha}\right]^{\nu_\mu}\left[\frac{s^2+a_1s+b_1}{(s+\alpha)^2}\right]^{\nu_1'}\left[\frac{s^2+a_2s+b_2}{(s+\alpha)^2}\right]^{\nu_2'}\cdots\left[\frac{s^2+a_ns+b_n}{(s+\alpha)^2}\right]^{\nu_n'}\cdot\frac{1}{(s+\alpha)^{q_\infty}}\cdot u(s)$$

In the above factorizations:

$$\boxed{\deg(n_u(s)) = \nu_1+\nu_2+\cdots+\nu_k+2\nu_1'+2\nu_2'+\cdots+2\nu_n' = \nu}$$

- The elements of the form:

$$\left\{\begin{array}{ll} \left[\dfrac{s+\lambda_i}{s+\alpha}\right] & \leftarrow g_i(s) \\[2ex] \left[\dfrac{s^2+a_is+b_i}{(s+\alpha)^2}\right] & \leftarrow g_j'(s) \\[2ex] \left[\dfrac{1}{(s+\alpha)^{q_\infty}}\right] & \leftarrow g_\infty(s) \end{array}\right.$$

Denote as

are called <u>primes</u> of $g(s)$ (modulo α)

- The set : $\left\{ (g_i(s))^{\nu_i}, (g_j'(s))^{\nu_j'}, (g_\infty(s))^{q_\infty} \right\}$

is the set of <u>elementary divisors</u> of $g(s) \in \Omega_{pr}^s$.

<u>Example</u>

$$g(s) = \frac{(s^2-s+1)(s-1)^2(s+2)}{(s+3)^2(s+4)^3(s+1)^2} \in \Omega_{pr}^s$$

$n_u(s) = (s^2-s+1)(s-1)^2$

q = degree diff bet. num. & den. + the degree of $n_u(s)$

$= 2+4 = 6$

$$g(s) = \frac{n_u(s)}{(s+\alpha)^q} \cdot \underbrace{\frac{n_s(s)(s+\alpha)^q}{d(s)}}_{u}$$

$\alpha > 0$

$$= \frac{(s^2-s+1)(s-1)^2}{(s+\alpha)^6} \cdot \underbrace{\frac{(s+2)(s+\alpha)^6}{(s+3)^2(s+4)^3(s+1)^2}}_{u(s)} \quad d(s)$$

$q_\infty = 2$

$$= \underbrace{\left[\frac{s-1}{s+\alpha}\right]^2 \left[\frac{s^2-s+1}{(s+\alpha)^2}\right]\left[\frac{1}{(s+\alpha)}\right]^2}_{\in\, \Omega_{pr}^s} u(s)$$

the associated primes are $\frac{s-1}{s+\alpha}$, $\frac{s^2-s+1}{(s+\alpha)^2}$, $\frac{1}{(s+\alpha)^2}$

Ω_{pr}^s

Let $g(s) \in \Omega^s_{pr}$ & define

$\pi(g(s))$: set of primes of $g(s)$

→ Let us consider $g_1(s) \in \Omega^s_{pr}$ & $g_2(s) \in \Omega^s_{pr}$

- $g_1(s)$ devides $g_2(s) \equiv \pi(g_1(s)) \subseteq \pi(g_2(s))$

→ Let $g_1(s)$ and $g_2(s)$ be the elemts of Ω^s_{pr}.

- $g_1(s)$ & $g_2(s)$ are <u>co-prime</u> in Ω^s_{pr} iff

$$\pi(g_1(s)) \cap \pi(g_2(s)) = \phi$$

→ Ex : $g_1(s) = \dfrac{s^2-1}{(s+2)^2(s+3)}$ $\qquad g_2(s) = \dfrac{s-1}{s+2}$

$$g_1(s) = \frac{(s-1)(s+1)}{(s+2)^2(s+3)}$$

$$= \frac{s-1}{(s+\alpha)^2} \cdot \underbrace{\frac{(s+1)(s+\alpha)^2}{(s+2)^2(s+3)}}$$

$q = 1+1$

$$= \left[\frac{s-1}{s+\alpha}\right]\left[\frac{1}{s+\alpha}\right] \cdot u(s)$$

$$\pi(g_1(s)) = \frac{s-1}{s+\alpha}, \quad \frac{1}{s+\alpha}$$

$$g_2(s) = \frac{s-1}{s+2} = \left[\frac{s-1}{s+\alpha}\right] \cdot \left[\frac{s+\alpha}{s+2}\right]$$

$$\pi(g_2(s)) = \frac{s-1}{s+\alpha}$$

$$\pi(g_1(s)) \cap \pi(g_2(s)) = \left[\frac{s-1}{s+\alpha}\right] \neq \phi$$

Hence $g_1(s)$ & $g_2(s)$ are not co-prime

Ex:- $g_1(s) = \dfrac{s+1}{s+2}$ $\qquad g_2(s) = \dfrac{1}{s+3} = \dfrac{1}{s+\alpha} \cdot \underbrace{\dfrac{s+\alpha}{s+3}}_{u(s)}$

$\pi(g_1(s)) = \phi$ $\qquad \pi(g_2(s)) = \dfrac{1}{s+\alpha}$

$g_1(s)$ & $g_2(s)$ are co-prime

→ $g_1(s) = \dfrac{(s+2)(s+3)}{(s+4)^2(s+1)}$ $\qquad g_2(s) = \dfrac{1}{s+3}$

$\downarrow \qquad\qquad \downarrow$

$\pi(g_1(s)) = \dfrac{1}{s+\alpha}$ $\qquad \pi(g_2(s)) = \dfrac{1}{s+\alpha}$

$$\pi(g_1(s)) \cap \pi(g_2(s)) = \frac{1}{s+\alpha}$$

Hence they are not co-prime.

→ $g_1(s) = \dfrac{s+2}{s+3}$ $\qquad g_2(s) = \dfrac{(s+3)(s+4)}{(s+5)(s+6)}$

$\pi(g_1(s)) = \phi$ $\qquad \pi(g_2(s)) = \phi$

When $g_1(s)$ & $g_2(s)$ are co-prime?

- If $g_1(s)$ & $g_2(s)$ are units in Ω^s_{pr} then they are co-prime
- If $g_1(s)$ is unit & $g_2(s)$ is strictly proper.
- Let $g_1(s)$ & $g_2(s)$ are biproper, then they are co-prime if there is no common unstable zeros.

- The rational functions $g_i(s) \in \Omega^s_{pr}$ are <u>co-prime</u> if and only if they do not have common unstable zeros and zeros at ∞.