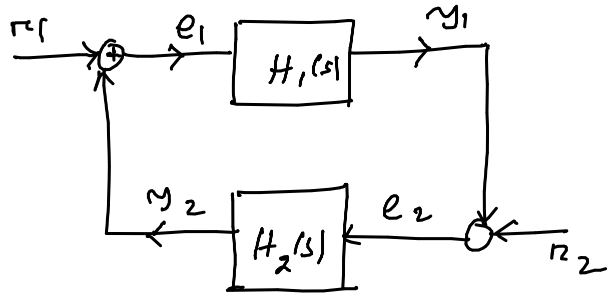
Lecture - 11



Let us assume that $H_1(s)$ is the transfer function of plant. $H_2(s)$ is the transfer function of controller, which needs to be designed to achieve internal stability in the FI.

$H_1(s)$ is given $\leftarrow$ t.f of plant

$H_2(s)$ is to be designed

$$\boxed{H_1(s) = \frac{b(s)}{a(s)} \quad \text{where} \quad \begin{array}{l} b(s) \in \mathbb{R}[s] \\ a(s) \in \mathbb{R}[s] \end{array}}$$

Assumptions :

(i) $H_1(s)$ is proper

(ii) Degree of $(a(s)) = n$

(iii) The polynomials $a(s)$ and $b(s)$ are co-prime.

The 4 t.f. which need to be stable

$$(1-H_1H_2)^{-1}, \quad (1-H_1H_2)^{-1}H_1, \quad (1-H_1H_2)^{-1}H_2, \quad (1-H_1H_2)^{-1}H_1H_2$$

Let us write $\boxed{H_2(s) = -\frac{y(s)}{x(s)}}$ $\quad y(s) \in \mathbb{R}[s]$, $x(s) \in \mathbb{R}[s]$

Assume that $H_2(s)$ is proper

and $\deg(x(s)) = m$

$$(1-H_1H_2)^{-1} = \left(1 + \frac{b(s)\,y(s)}{a(s)\,x(s)}\right)^{-1} = \frac{a(s)\,x(s)}{a(s)\,x(s) + b(s)\,y(s)}$$

$$(1-H_1H_2)^{-1}H_1 = \frac{b(s)\,x(s)}{a(s)\,x(s) + b(s)\,y(s)}$$

$$(1-H_1H_2)^{-1}H_2 = \frac{-a(s)\,y(s)}{a(s)\,x(s) + b(s)\,y(s)}$$

$$(1-H_1H_2)\,H_1H_2 = \frac{-b(s)\,y(s)}{a(s)\,x(s) + b(s)\,y(s)}$$

So the closed loop characteristic polynomial

is $\boxed{\sigma(s) = a(s)\,x(s) + b(s)\,y(s)}$

$\deg(\sigma(s)) = ? \rightarrow m+n$

To design $H_2(s)$, s.t. the closed loop characteristic polynomial $\sigma(s)$ is stable, we choose a set of $(n+m)$ complex numbers λ_i s.t. $Re(\lambda_i) < 0$. (include its conjugate)

Then construct $\sigma(s) = (s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_{n+m})$

write it as

$$\sigma(s) = \sigma_{n+m} s^{n+m} + \sigma_{n+m-1} s^{n+m-1} + \cdots + \sigma_1 s + \sigma_0$$

$$\sigma_i \in \mathbb{R}$$

$$\boxed{a(s)\, x(s) + b(s)\, y(s) = \sigma(s)} \leftarrow \text{Diophantine Equation}$$

Given $a(s)$, $b(s)$ & $\sigma(s)$, compute $x(s)$ & $y(s)$.

Ex: $H_1(s) = \dfrac{b_1(s) + b_0}{a_2 s^2 + a_1 s + a_0}$ $\qquad H_2(s) = \dfrac{y_1 s + y_0}{x_1 s + x_0}$

$$a(s)\, x(s) + b(s)\, y(s) = (a_2 s^2 + a_1 s + a_0)(x_1 s + x_0) + (b_1 s + b_0)(y_1(s) + y_0)$$

$$= a_2 x_1 s^3 + (a_2 x_0 + a_1 x_1 + b_1 y_1) s^2 + (a_1 x_0 + a_0 x_1 + b_1 y_0 + b_0 y_1) s + a_0 x_0 + b_0 y_0 \quad \ldots (*)$$

$$\sigma(s) = \sigma_3 s^3 + \sigma_2 s^2 + \sigma_1 s + \sigma_0 \quad \ldots (**)$$

Now comparing (*) & (**) (equating the coefficients)

$$\begin{cases} a_0 x_0 + b_0 y_0 = \sigma_0 \\ a_1 x_0 + a_0 x_1 + b_1 y_0 + b_0 y_1 = \sigma_1 \\ a_2 x_0 + a_1 x_1 + b_1 y_1 = \sigma_2 \\ a_2 x_1 = \sigma_3 \end{cases}$$

$$\underbrace{\begin{bmatrix} a_0 & 0 & b_0 & 0 \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & 0 & b_1 \\ 0 & a_2 & 0 & 0 \end{bmatrix}}_{\text{known}} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ y_0 \\ y_1 \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}}_{\text{known}} \qquad A x = b$$

Sylvester matrix

$$H_1(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

(Assume $H_1(s)$ to be strictly proper)

The controller is represented as

$$H_2(s) = \frac{y(s)}{x(s)} = \frac{y_m s^m + \cdots y_1 s + y_0}{x_m s^m + \cdots + x_1 s + x_0}$$

corresponding to the polynomials $a(s)$ & $b(s)$

let us define a matrix $R(a,b)$ of

size $(n+m+1) \times 2(m+1)$:

$$R(a,b) = \left[\begin{array}{cccc|cccc} a_0 & 0 & & O & b_0 & 0 & & O \\ a_1 & a_0 & & & b_1 & b_0 & & \\ \vdots & \vdots & \ddots & & \vdots & b_1 & \ddots & \\ a_{n-1} & a_{n-2} & \cdots & a_0 & b_{n-1} & b_{n-2} & \cdots & b_0 \\ a_n & a_{n-1} & \cdots & a_1 & 0 & b_{n-1} & \cdots & b_1 \\ 0 & a_n & & \vdots & 0 & 0 & & \vdots \\ & & & a_{n-1} & \vdots & \vdots & & b_{n-1} \\ & & & a_n & 0 & 0 & & 0 \end{array}\right]$$

$m+1$ col. $\qquad\qquad$ $m+1$ col.

Sylvester matrix

Corresponding the polynomials $x(s)$ & $y(s)$

define the following coefficient vector

$$k = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \\ \hdashline y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}_{2(m+1)} \qquad \sigma = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{n+m-1} \\ \sigma_{n+m} \end{bmatrix}$$

from the eq^n $a(s)\,x(s) + b(s)\,y(s) = \sigma$

can be represented $\boxed{[R(a,b)]k = \sigma}$

Hence to design $H_2(s) = \frac{y(s)}{x(s)}$, we have to solve the set of linear equation, which is compactly written as

$$\boxed{\left[R(a,b)\right] k = \sigma}$$ Solve for k

For $\underline{m = n-1}$, what is the size of $R(a,b) = 2n \times 2n$

square matrix

- Since $a(s)$ & $b(s)$ are co-prime, the Sylvester matrix $\left[R(a,b)\right]$ is non-singular.

then

$$\boxed{k = \left[R(a,b)\right]^{-1} \sigma}$$

For

(i) $\underline{m = n}$, $\text{Size}\left(\left[R(a,b)\right]\right) = (2n+1) \times (2n+2)$

fat matrix

If $a(s)$ & $b(s)$ are co-prime then

$R(a,b)$ is full row rank matrix.

(ii) $m = n-2$, $\text{Size}\left(\left[R(a,b)\right]\right) = (2n-1) \times (2n-2)$

tall matrix

If $a(s)$ & $b(s)$ are co-prime then

$R(a,b)$ is full-column rank matrix.

The order of the controller is "m"
order of a plant is "n"

For $m = n-1$

$R(a,b)$ is square & non-singular
& hence $H_2(s)$ exists.
s.t. the FI is internally stable.

For $m \geq n-1$, $R(a,b)$ is fat matrix,
& is of full row rank
$\Rightarrow$ controller always exists s.t FI is internally stable.

$m < n-1$, $R(a,b)$ is tall, & hence.
the controller may not exist.

→ The roots of

$\sigma(s) = a(s)\,x(s) + b(s)\,y(s)$ are the closed loop poles.

- For arbitrary pole assignment the minimum order controller for an n^{th} order plant is $m = n-1$.

Ex : $H_1(s) = \dfrac{s-1}{s(s-2)}$. $a(s) = s^2 - 2s$
$b(s) = s - 1$

We need to design $H_2(s)$ to place the closed loop poles at $-1, -2, -3$

then $\sigma(s) = s^3 + 6s^2 + 11s + 6$

$$\sigma = \begin{bmatrix} 6 \\ 11 \\ 6 \\ 1 \end{bmatrix}$$

$$R(a,b) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -2 & 0 & 1 & -1 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$k = \left[R(a,b)\right]^{-1}\sigma = \begin{bmatrix} -25 \\ 1 \\ -6 \\ 33 \end{bmatrix}$$

$$H_2(s) = \frac{33s - 6}{s - 25}$$

So the controller for making the FI stable

is $H_2(s) = -\dfrac{33s - 6}{s - 25}$

The designed $H_2(s)$ is called stabilizing controller for the FI.

- We will show that the designed controller does not have unstable pole - zero cancellation.

Let partition the polynomial $\sigma(s)$ as follows:

$$\sigma(s) = \underbrace{\sigma_n(s)}_{\substack{\uparrow \\ n\text{th} \\ \text{deg.}}} \underbrace{\sigma_m(s)}_{\substack{\uparrow \\ m\text{th degree}}}$$

$$H_1(s) = \frac{B(s)}{A(s)} = \frac{b(s)/\sigma_n(s)}{a(s)/\sigma_n(s)}$$

$$H_2(s) = -\frac{Y(s)}{X(s)} = -\frac{y(s)/\sigma_m(s)}{x(s)/\sigma_m(s)}$$

In this framework $\{A(s), B(s), X(s), Y(s)\} \in \Omega^{s}_{pr}$

$\{b(s), a(s), x(s), y(s)\} \in \mathbb{R}[s]$

$\left\{ \begin{array}{l} A(s) \ \& \ X(s) \text{ are biproper} \\ Y(s) \ \& \ B(s) \text{ are proper} \end{array} \right.$

$$(1 - H_1 H_2)^{-1} = \frac{1}{AX + BY} \cdot AX$$

- $(1 - H_1 H_2)^{-1} H_1 H_2 = -\frac{1}{AX+BY} \cdot BY$

$$(1 - H_1 H_2)^{-1} H_1 = \frac{1}{AX + BY} \cdot BX$$

$$(1 - H_1 H_2) H_2 = -\frac{1}{AX + BY} \cdot AY$$

The elements $\{AX, BX, -AY, -BY\} \in \Omega^s_{pr}$

For stability of all these 4 t.f.

we need $\frac{1}{AX+BY}$ must be stable & proper.

$$\frac{1}{AX+BY} = \frac{\sigma_n \sigma_m}{ax+by} \begin{matrix} \leftarrow \text{deg } n+m \\ \leftarrow \text{deg } n+m \end{matrix}$$

↑ is a biproper rational function.

↳ To make $\frac{1}{AX+BY}$, stable the

polynomial $ax+by$ must be

stable.

- The 4 t.fs are proper & stable if and only if $AX+BY$ is a unit in Ω^s_{pr}.

$$\boxed{AX+BY = \frac{ax+by}{\sigma_n \sigma_m} = 1}$$

$x(s)$ & $y(s)$ are designed in such a manner

that $ax+by = \sigma(s) = \sigma_n \sigma_m$

$A(s)$ & $B(s)$ are co-prime in $\underline{\underline{\Omega^s_{pr}}}$.

$$\begin{matrix} A(s) & & B(s) \\ \downarrow & & \downarrow \\ \frac{a(s)}{\sigma_m(s)} & & \frac{b(s)}{\sigma_n(s)} \end{matrix}$$

↑ Since $a(s)$ & $b(s)$ are co prime

We have designed $X(s)$ & $Y(s)$

from $AX + BY = 1$

$$X(s) = \frac{x(s)}{\sigma_m(s)} \leftarrow \text{biproper}$$

$$Y(s) = \frac{y(s)}{\sigma_m(s)} \leftarrow \text{proper}$$

Hence $X(s)$ & $Y(s)$ do not have common zeros at ∞.

- $X(s)$ & $Y(s)$ also do not have common unstable zeros.

$\downarrow$

Proof   Assume that λ_0 is an unstable zero of $X(s)$ & $Y(s)$.

$$(s+\lambda_0)\left[a(s)\tilde{x}(s) + b(s)\tilde{y}(s)\right] = \sigma(s) \quad — (*)$$

$$x(s) = (s+\lambda_0)\,\tilde{x}(s)$$

$$y(s) = (s+\lambda_0)\,\tilde{y}(s)$$

$\sigma(s)$ also has a root at λ_0

$$\sigma(\lambda_0) = 0$$

However, the roots of $\sigma(s)$ are in the open left half of complex plane.

Hence $X(s)$ & $Y(s)$ are <u>co-prime</u> in Ω^{s}_{pr}'

Hence there is no unstable pole-zero cancellation.

The set of stabilizing controllers:

Let say we have designed

A particular sol^n

$X'(s)$ & $Y'(s)$ } ← by solving the Diophantine eq^n

$$a(s)x(s) + b(s)y(s) = \sigma(s)$$

The stabilizing controllers are

$$\boxed{H_2(s) = -\frac{Y'(s) - A(s)Q(s)}{X'(s) + B(s)Q(s)}}$$

← Yula-Kucera parameterization.

when $Q(s) \in \Omega^{s}_{pr}$ s.t. $X'(s) + B(s)Q(s) \neq 0$.