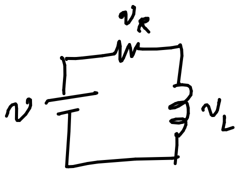
Lecture - 12

UNCERTAINTY & ROBUSTNESS



$$\frac{di}{dt} = -\frac{R}{L} i + \frac{1}{L} v$$

Resistor is an uncertain parameter

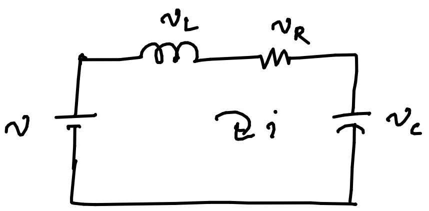
↳ comes with some tolerance

$5\,\Omega \pm 0.1\%$

$R \to 5 \longrightarrow R + \Delta R$

$$\dot{x} = (a+\delta)x + bu$$

$\delta \to$ uncertain parameter



$$\frac{di}{dt} = -\frac{R}{L} i - \frac{1}{L} v_c + \frac{1}{L} v$$

$$\frac{dv_c}{dt} = \frac{1}{C} i$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v.$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ v_c \end{bmatrix}$$

Assume that L & C are fixed & accurate & uncertainty appears in the resistor.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} + \underline{\underline{\delta}} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} v$$

The characteristic polynomial

$$p(s) = s^2 + \frac{R}{L} s + \frac{1}{LC} \qquad (\delta = 0)$$

$$= s^2 + as + b$$

$$p(s, \delta) = s^2 + a(\delta) s + b \qquad \delta \in [\delta^-, \delta^+]$$

$a(\delta)$ is uncertain coefficient.

→ Ex. of Vertical take-off and Landing helicopter

$$A(\delta) = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 + \delta_1 & -0.7070 & 1.4200 + \delta_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B(\delta) = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 + \delta_3 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} |\delta_1| &\leq 0.05 \\ |\delta_2| &\leq 0.01 \\ |\delta_3| &\leq 0.01 \end{aligned}$$

$$C(\delta) = [0 \quad 1 \quad 0 \quad 0]$$

δ is an uncertain parameter & it depends on change in air speed.

$$p(s,\delta) = s^4 + 1.754\, s^3 + (-0.0024\,\delta_1 - \delta_2 - 0.6472)\, s^2$$
$$+ (0.0624 + 4.02\,\delta_1 - 1.047\,\delta_2)\, s$$
$$+ (0.1698\,\delta_1 - 0.0356\,\delta_2 + 0.686)$$

$$p(s,\delta) = p_4(\delta)\, s^4 + p_3(\delta)\, s^3 + p_2(\delta)\, s^2 + p_1(\delta)\, s + p_0(\delta)$$

$$\delta \in [\delta^-, \delta^+]$$

$$\left.\begin{array}{l} \begin{cases} \dot{x} = A(\delta)\, x + B(\delta) u \\ y = C(\delta)\, x \end{cases} \\ \{\, G(s,\delta) = C(\delta)\,(sI - A(\delta))^{-1} B(\delta) \end{array}\right\}$$

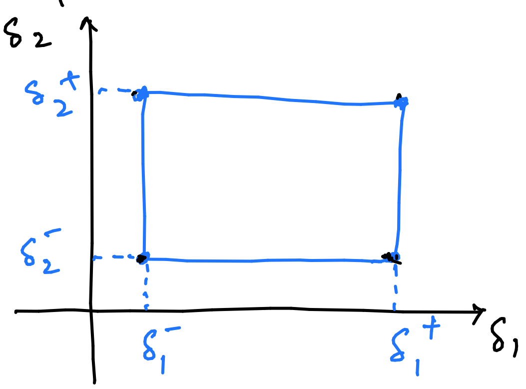
Representation of uncertain plant model.

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_\ell \end{bmatrix} \qquad \delta_i \in [\delta_i^-, \delta_i^+]$$

$$\Delta := \left\{ \delta = [\delta_1 \ \delta_2 \cdots \delta_\ell]^T : \ \delta_i \in [\delta_i^-, \delta_i^+] \ \text{for } i = 1, 2 \cdots \ell \right\}$$

Uncertain parameter set

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$



$$\delta_1 \in [\delta_1^-, \delta_1^+]$$
$$\delta_2 \in [\delta_2^-, \delta_2^+]$$

Δ is a Hyperrectangle in $\mathbb{R}^\ell$.
(a special case of polytope)

- Vertex of Δ :

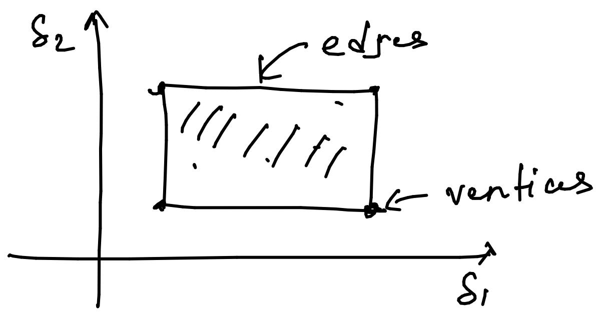
The vertices are obtained by setting each δ_i to δ_i^+ or δ_i^-

$$\delta_V := \left\{ \delta \; : \; \delta_i = \delta_i^- \text{ or } \delta_i^+ \text{ for } i = 1, 2 \cdots l \right\}$$

- Exposed Edges :

For a fixed 'i', an edge is the set

$$\Delta_{E_i} := \left\{ \delta \; : \; \delta_i^- \le \delta_i \le \delta_i^+ , \;\; \delta_j = \delta_j^- \text{ or } \delta_j^+ \text{ for all } i \neq j \right\}$$



Exposed edges :

$$\Delta_E = \bigcup_{i=1}^{l} \Delta_{E_i}$$

Let $\quad \delta = [\delta_1 \ \delta_2 \ \cdots \ \delta_\ell]^T \qquad \delta_i \in [\delta_i^-, \delta_i^+]$

$$\Delta = \{ \delta : \delta = [\delta_1 \ \delta_2 \cdots \delta_\ell]^T, \ \delta_i \in [\delta_i^-, \delta_i^+] \}$$

Uncertain polynomial:

$$p(s,\delta) = p_0(\delta) + p_1(\delta) s + p_2(\delta) s^2 + \cdots + p_n(\delta) s^n$$

when $\delta \in \Delta$

$$P(s,\Delta) := \{ p(s,\delta) : \delta \in \Delta \}$$

family of uncertain polynomials

$$p(s) = p_0 + p_1 s + \cdots + p_n s$$

Roots $[p(s)] \in \mathbb{C}^-$ (open left half of complex plane)

then $p(s)$ is stable.

- Stability of <u>family of polynomials</u>: $P(s,\Delta)$

$$\text{Roots}\,[P(s,\Delta)] := \{ s_0 \in \mathbb{C} : p(s_0,\delta) = 0 \ \& \ p(s,\delta) \in P(s,\Delta) \}$$

- The polynomial family is "<u>robustly stable</u>" iff the root set Roots $[P(s,\Delta)]$ contained in $\mathbb{C}^-$.

The root set Roots $[P(s,\Delta)]$ contains infinite complex numbers. Hence,

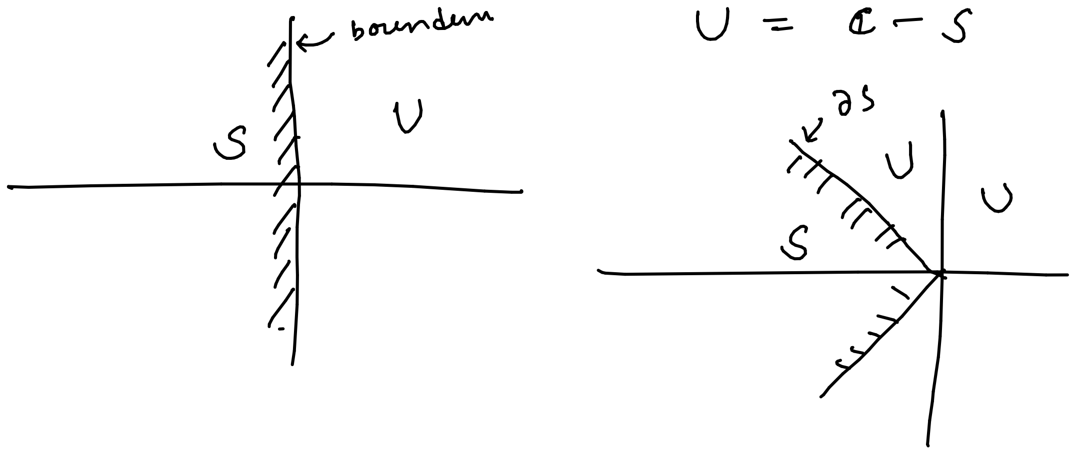
- Checking the robust stability of a family of polynomials $P(s,\Delta)$ is <u>not easy</u>.

→ <u>Boundary Crossing Theorem:</u>

$S \subset \mathbb{C}$ S is an open set in $\mathbb{C}$

∂S is the boundary of S.

$U = \mathbb{C} - S$



Consider a family of polynomials $P(s,\Delta)$ where the elements are of the following form:

$$p(s,\delta) = p_0(\delta) + p_1(\delta)s + p_2(\delta)s^2 + \dots + p_n(\delta)s^n$$

& they satisfy the following assumptions:

i) $P(s,\Delta)$ is a family of polynomials of <u>fixed degree</u> 'n', i.e. ($p_n(\delta) \neq 0$ for interval $[\delta^-, \delta^+] \in \Delta$)

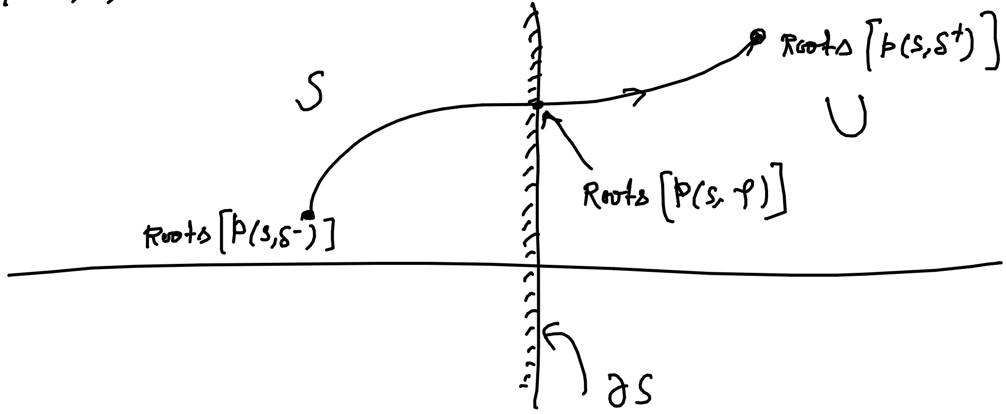
ii) they are continuous w.r.t. δ on a fixed interval $[\delta^-, \delta^+] \in \Delta$, i.e. their coefficients $p_i(\delta)$ are continuous funct^n of δ over interval: $[\delta^-, \delta^+]$.

- Result :

With the above assumptions, suppose that $p(s, \delta^-)$ has all its roots in S whereas $p(s, \delta^+)$ has at least one root in U. Then there exist at least one φ in $(\delta^-, \delta^+]$ such that

(i) $p(s, \varphi)$ has all its roots in $S \cup \partial S$,

(ii) $p(s, \varphi)$ has at least one root in ∂S.



- Intuitively B.C.T. says that going from stable open set S to another open set, which is unstable U, the root set of a continuous family of polynomials $P(s, \Delta)$, of fixed degree, must intersect the boundary (∂S) of stable open set S at some intermediate stage.

(the root set can not jump from one open set S to another disjoint open set U)

- Result

Let the family of polynomials be $P(s, \Delta)$.
If there exists $\delta^1 \in \Delta$ & $\delta^2 \in \Delta$ s.t.

$$p(s, \delta^1) = p_0(\delta^1) + p_1(\delta^1) s + \cdots + p_n(\delta^1) s^n$$

is stable, and

$$p(s, \delta^2) = p_0(\delta^2) + p_1(\delta^2) + \cdots + p_n(\delta^2) s^n$$

is unstable.

Then the roots of $P(s, \Delta)$ contains at least one point on the non-negative $j\omega$-axis i.e. there exists $\omega \geq 0$ s.t. $j\omega \in \text{Roots}[P(s, \Delta)]$.

→ Boundary Crossing Theorem due to Frazer & Duncan:

Let $P(s, \Delta)$ be a family of polynomials.
The set $P(s, \Delta)$ is robustly stable if and only if

(i) there exists a stable polynomial $p(s, \delta) \in P(s, \Delta)$

(ii) $j\omega \notin \text{Roots}[P(s, \Delta)]$ for all $\omega \geq 0$.