Lecture - 13

- Hurwitz matrices for robust stability analysis of a family of uncertain polynomials:

Consider a polynomial:

$$p(s) = p_0 + p_1 s + \cdots + p_n s^n$$ (not an uncertain polynomial)

The associated Hurwitz matrices:

$$H_1 = p_{n-1}$$

$$H_2 = \begin{bmatrix} p_{n-1} & p_{n-3} \\ p_n & p_{n-2} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} p_{n-1} & p_{n-3} & p_{n-5} \\ p_n & p_{n-2} & p_{n-4} \\ 0 & p_{n-1} & p_{n-3} \end{bmatrix}$$ → This pattern continues untill $n \times n$ matrix is obtained.

For a 4th order polynomial

$$H_n = H_4 = \begin{bmatrix} p_3 & p_1 & 0 & 0 \\ p_4 & p_2 & p_0 & 0 \\ 0 & p_3 & p_1 & 0 \\ 0 & p_4 & p_2 & p_0 \end{bmatrix}$$

→ for $n =$ even

For 5th degree polynomial

$$H_5 = \begin{bmatrix} p_4 & p_2 & p_0 & 0 & 0 \\ p_5 & p_3 & p_1 & 0 & 0 \\ 0 & p_4 & p_2 & p_0 & 0 \\ 0 & p_5 & p_3 & p_1 & 0 \\ 0 & 0 & p_4 & p_2 & p_0 \end{bmatrix}$$

$n =$ odd

→ Result

The polynomial $p(s)$ is stable if and only if

$$\det(H_i) > 0 \quad \text{for} \quad i = 1, 2 \cdots n.$$

- The same result can be used for robust stability of a family of uncertain polynomials.

$$p(s,\delta) = p_0(\delta) + p_1(\delta)s + \cdots + p_n(\delta)s^n \quad (\delta \in \Delta)$$

$$H_1(\delta) = p_{n-1}(\delta)$$

$$H_2(\delta) = \begin{bmatrix} p_{n-1}(\delta) & p_{n-3}(\delta) \\ p_n(\delta & p_{n-2}(\delta) \end{bmatrix}$$

⋮

Continue this way upto $H_n(\delta)$.

- For robust stability all of the Hurwitz matrices

$$\det[H_i(\delta)] > 0 \quad \text{for } \delta \in \Delta$$

Ex: $\delta = \begin{bmatrix} m_L \\ l \end{bmatrix}$ $\quad m_L \in [50, 2395]$ (Crane model)
$\quad l \in [7, 12]$ (book Ackermann)

$$p(s,\delta) = \frac{6}{l} + \frac{20}{l}s + \frac{0.6l + 20 + 0.01m_L}{l}s^2 + 2s^3 + s^4$$

$$H_1(s) = 2$$

$$H_2(s) = \begin{bmatrix} 2 & \frac{20}{l} \\ 1 & \frac{0.6l + 20 + 0.01m_L}{l} \end{bmatrix} \rightarrow \det \rightarrow \frac{1000 + 60l + m_L}{50l}$$

$$H_3(s) = \begin{bmatrix} 2 & \frac{20}{l} & 0 \\ 1 & \frac{0.6l + 20 + 0.01m_L}{l} & \frac{6}{l} \\ 0 & 2 & \frac{20}{l} \end{bmatrix} \rightarrow \det(H_3(s)) = \frac{2000 + 2m_L}{5l^2}$$

$$\det H_4(s) = \frac{6}{l} \cdot \det(H_3(s))$$

- Observe that the $\det(H_i(s)) > 0$ for $i = 1, 2 \cdots 4$ for positive values of m_L & l.

Hence the uncertain polynomial $p(s, \delta)$ is robustly stable.

→ Frazer & Duncan Result:

Let $P(s, \Delta)$ be a family of uncertain polynomials. Then the set $P(s, \Delta)$ is robustly stable if and only if:

(i) there exists a stable polynomial $p(s, \delta) \in P(s, \Delta)$,

✓ (ii) $\det(H_n(s)) \neq 0$ for all $\delta \in \Delta$.

- Internal Polynomials

Let us write an uncertain polynomial in the following form (Interval polynomial):

$$p_I(s,\delta) = \delta_0 + \delta_1 s + \delta_2 s^2 + \dots + \delta_{n-1} s^{n-1} + \delta_n s^n$$

with uncertain coefficients $\delta = [\delta_0\ \delta_1\ \cdots\ \delta_n]^T$

$\delta_i \in [\delta_i^-, \delta_i^+]$. —— (*)

Previous cases

$$p(s,\delta) = p_0(\delta) + p_1(\delta)s + \cdots + p_n(\delta)s^n$$

$p_i(\delta)$ are continuous functions of δ. —— (**)

For (*), the coefficients vary within a box Δ (the variation is independent to each other)

(**), the coefficients may not vary within a box, whereas the uncertain parameter δ belongs to a box Δ.

- The polynomial $p_I(s,\delta)$ of the form (*) is called internal polynomial, if δ varies over the box

$$\Delta := \{ \delta \mid \delta_i \in [\delta_i^-, \delta_i^+] \}$$
for $i = 1, 2 \cdots n$

- For internal polynomials, the coefficients vary independently.

Ex: $p(s,\delta) = p_0(\delta) + p_1(\delta)s + p_2(\delta)s^2 + p_3(\delta)s^3$

where

$$\left.\begin{aligned} p_0(\delta) &= 1 \\ p_1(\delta) &= 3 - 2\delta_1 - 0.5\delta_2 \\ p_2(\delta) &= 0.5 + \delta_1 + 1.5\delta_2 \\ p_3(\delta) &= 1 \end{aligned}\right\} \quad \delta_i \in [0,1]$$

Since the coefficients are not mutually independent the above uncertain polynomial is not an interval polynomial.

- Kharitonov's Result

(for interval polynomials $p_I(s,\delta)$)

- Consider a family of uncertain polynomials

$$P_I(s,\Delta) := \left\{ p_I(s,\delta) = \delta_0 + \delta_1 s + \dots + \delta_n s^n \;\middle|\; \delta \in \Delta \text{ and } \delta_n > 0,\ \delta_i \in [\delta_i^-, \delta_i^+] \right\}$$

Let us define following four extreme polynomials (called Kharitonov's polynomials)

$$\begin{cases} p^{+-}(s) = \delta_0^+ + \delta_1^- s + \delta_2^- s^2 + \delta_3^+ s^3 + \delta_4^+ s^4 + \dots \\ p^{++}(s) = \delta_0^+ + \delta_1^+ s + \delta_2^- s^2 + \delta_3^- s^3 + \delta_4^+ s^4 + \dots \\ p^{-+}(s) = \delta_0^- + \delta_1^+ s + \delta_2^+ s^2 + \delta_3^- s^3 + \delta_4^- s^4 + \dots \\ p^{--}(s) = \delta_0^- + \delta_1^- s + \delta_2^+ s^2 + \delta_3^+ s^3 + \delta_4^- s^4 + \dots \end{cases}$$

- Result

The interval polynomial family $P_I(s, \Delta)$ is robustly stable if and only if all of the 4 Kharitonov's polynomials are stable.

Consider $v_1, v_2 \cdots v_r$ be the set of points in $\mathbb{R}^n$. For $\lambda_i \in \mathbb{R}$

- $\sum_{i=1}^{r} \lambda_i v_i \leftarrow$ linear combination of points $v_1, v_2 \cdots v_r$ in $\mathbb{R}^n$

- Let $\lambda_i \geq 0$, then

$\sum_{i=1}^{r} \lambda_i v_i \leftarrow$ non-negative combination of points $v_1, v_2 \cdots v_r$ (conic combination)

- Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$, then

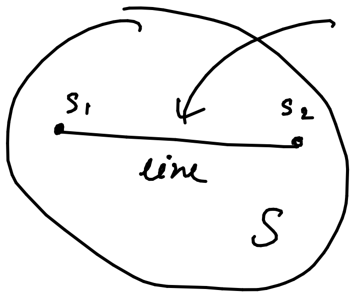
$\sum_{i=1}^{r} \lambda_i v_i \leftarrow$ affine combination of points $v_1, v_2 \cdots v_r$

- Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$ & $\lambda_i \geq 0$ for $\forall i$, then

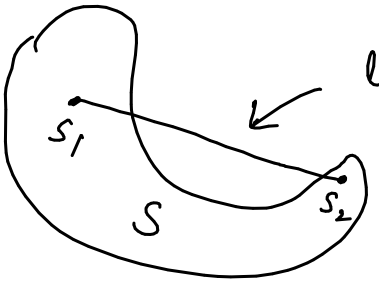
$\sum_{i=1}^{r} \lambda_i v_i \leftarrow$ convex combination of points $v_1, v_2 \cdots v_r$

- **Convex Set**: A set S is a convex set if the line, joining between any two points s_1 & s_2, in S remains in S.

$(1-\lambda)s_1 + \lambda s_2$ for $\lambda \in [0,1]$



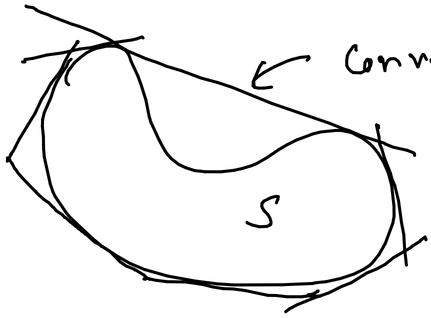
← convex set



line is not inside S, hence the set is not convex.

- Given a set S, which is not necessarily a convex set, its <u>convex hull</u> is the smallest convex set which contains S.

Convex hull of a set S is the set of all convex combinations of points in S i.e.



← Convex hull of S

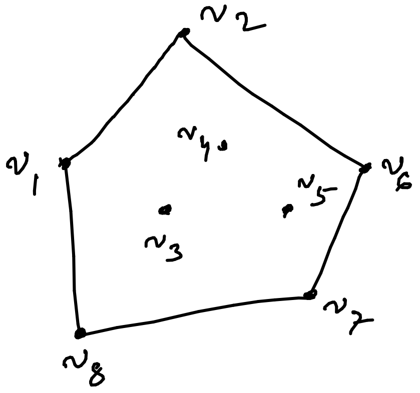
$$Co(S) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \; : \; \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1 \text{ and } \lambda_i \geq 0 \text{ for } i = 1, 2 \ldots n \right\}$$

- A <u>polytope</u> P is the convex hull of a finite set of points.

P = convex hull $\{ v_1, v_2 \cdots v_n \}$

$\downarrow$

Conv$\{ v_1, v_2, \cdots, v_n \}$ or Co$\{ v_1, v_2, \cdots, v_n \}$

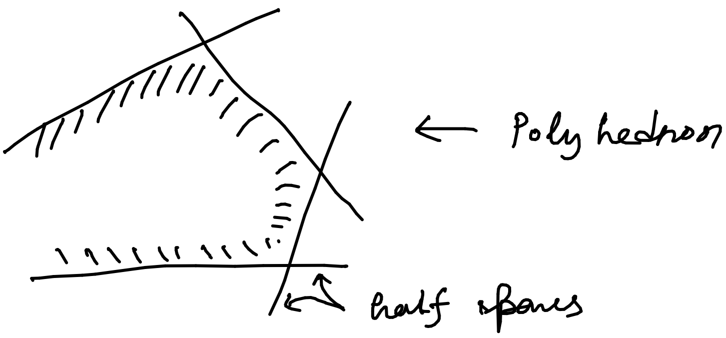


The convex hull of the points $v_1, v_2 \cdots v_8$ is generated by points v_1, v_2, v_6, v_7 & v_8.

vertices of P

Polytope P = convex hull $\{ v_1, v_2, v_6, v_7, v_8 \}$.

- <u>Polyhedron</u> is the intersection of finite number of half spaces i.e. $\{ x : a_i^T x \le b_i$ for $i = 1, 2 \cdots k$



- A <u>bounded</u> polyhedron is a <u>polytope</u>.