Lecture - 14

• Polytope of polynomials:

→ Consider a vector $\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_l \end{bmatrix}$ where δ_i are uncertain parameters.

Let us define a set

$$\Delta = \{ \delta \in \mathbb{R}^l : \delta_i \in [\delta_i^-, \delta_i^+] \}$$

It is a polytope

Vertex set of Δ:

$$\Delta_V = \{ \delta \mid \delta_i = \delta_i^- \text{ or } \delta_i = \delta_i^+ \text{ for } i = 1, 2 \cdots l \}$$

• For a fixed i:

$$E_i = \{ \delta \mid \delta_i^- \leq \delta_i \leq \delta_i^+, \ \delta_j = \delta_j^+ \text{ or } \delta_j = \delta_j^- \text{ for all } j \neq i \}$$

• The exposed edges of Δ:

$$\Delta_E = \bigcup_{i=1}^{l} E_i$$

Let us define a set $P_\ell(s, \Delta)$ where the elements $p(s,\delta) \in P_\ell(s,\Delta)$ has the following form:

$$p(s,\delta) = p_0(\delta) + p_1(\delta)s + \cdots + p_{n-1}(\delta)s^{n-1} + p_n(\delta)s^n$$

where the coefficients satisfy following properties:

(i) $p_i(\delta)$ are (affine) linear functions of uncertain parameter δ.

(ii) The uncertain parameter $\delta \in \Delta$.

Then, we will say $P_\ell(s,\Delta)$ is a polytopic family of polynomials or we can say $P_\ell(s,\Delta)$ is a polytope of polynomials.

$$P_\ell(s,\Delta) := \left\{ p(s,\delta) = p_0(\delta) + p_1(\delta)s + \cdots + p_n(\delta)s^n \text{ such that } p_i(\delta) \text{ are linear in } \delta \text{ and } \delta \in \Delta \right\}$$

polytope of polynomials

Ex: $p(s,\delta) = \underbrace{(2\delta_1 - \delta_2 + 5)}_{p_0(\delta)} + \underbrace{(4\delta_1 + 3\delta_2 + 2)}_{p_1(\delta)}s + s^2$

$\delta = \begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}$ $\qquad |\delta_i| \leq 1$

The <u>vertex</u> set of Polytope $P_\ell(s,\Delta)$:

$$P_\ell(s,\Delta_V) = \{ p(s,\delta) \;:\; \delta \in \Delta_V \}$$

The Exposed <u>edge</u> set of polytope $P_\ell(s,\Delta)$

$$P_\ell(s,\Delta_E) = \{ p(s,\delta) \;:\; \delta \in \Delta_E \}$$

Let us write the elements of

$P_\ell(s,\Delta_V)$ as $\underline{p(s,\delta^{v_i})}$

Then the polytope $P_\ell(s,\Delta)$ can be described as follows :

- Since $p(s,\delta^{v_i})$ are vertices of $P_\ell(s,\Delta)$,

  $P_\ell(s,\Delta)$ is :

  convex hull

$$P_\ell(s,\Delta) = \mathrm{conv}\{ p(s,\delta^{v_1}), p(s,\delta^{v_2}), \dots, p(s,\delta^{v_k}) \}$$

k^{th} vertex polynomial

Then each element $p(s,\delta) \in P_\ell(s,\Delta)$ can be generated as follows :

$$p(s,\delta) = \sum_{i=1}^{k} \lambda_i \, p(s,\delta^{v_i}) \quad \text{where } \sum_{i=1}^{k} \lambda_i = 1 \;\&\; \lambda_i \geq 0$$

convex combinations of vertex polynomials

$$p(s,\delta) = (2\delta_1 - \delta_2 + 5) + (4\delta_1 + 3\delta_2 + 2)s + s^2$$

$$|\delta_i| \leq 1$$

So The vertex points of Δ (or vertex polynomials)

$$\delta^{v_1} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \delta^{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \delta^{v_3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \delta^{v_4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the vertex points of $P_e(s,\Delta)$ are:

$$p(s,\delta^{v_1}) = 4 - 5s + s^2 \qquad p(s,\delta^{v_2}) = 2 + s + s^2$$

$$p(s,\delta^{v_3}) = 8 + 3s + s^2 \qquad p(s,\delta^{v_4}) = 6 + 9s + s^2$$

then

$$P_e(s,\Delta) = \operatorname{conv}\left\{ p(s,\delta^{v_1}),\ p(s,\delta^{v_2}),\ p(s,\delta^{v_3}),\ p(s,\delta^{v_4}) \right\}$$

Then each element in $P_e(s,\Delta)$ can be represented as follows..

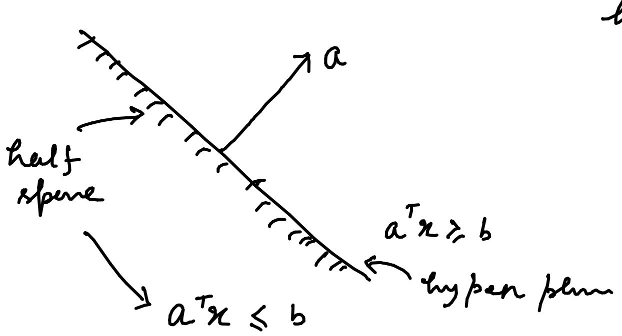
$$p(s,\delta) = \lambda_1 p(s,\delta^{v_1}) + \lambda_2 p(s,\delta^{v_2}) + \lambda_3 p(s,\delta^{v_3}) + \lambda_4 p(s,\delta^{v_4})$$

where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$

& $\lambda_i \geq 0$ for $i = 1, 2, 3, 4$.

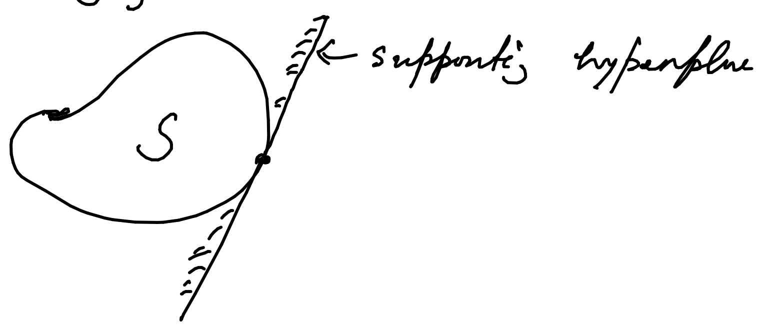
→ Hyperplane : $\{ x \mid a^T(x - x_0) = 0 \}$

$a \neq 0$ which is orthogonal to the hyperplane.



- Supporting hyper plane of a set S :

It is a hyperplane that contains S in one of its half space and intersect the boundary of S with at least at one point.



→ The supporting hyper plane $\mathcal{H}$ of $P_\ell(s,\Delta)$ is a hyper plane, which contains $P_\ell(s,\Delta)$ in one of its half space & intersect the boundary of $P_\ell(s,\Delta)$ with at least one point.

→ The <u>exposed sets</u> of $P_\ell(s,\Delta)$ are consisting of those points which are in the set

$$\underbrace{\mathcal{H}}_{\text{convex set}} \cap \underbrace{P_\ell(s,\Delta)}_{\text{convex set}}$$

- Since intersection of convex sets are also convex, the exposed sets : $\mathcal{H} \cap P_\ell(s,\Delta)$ is a convex set.

- The one dimensional exposed sets are called exposed edges of $P_\ell(s,\Delta)$.

- The two dimensional exposed sets are called exposed surfaces of $P_\ell(s,\Delta)$.

→ Corresponding to the uncertain polynomial

$$p(s,\delta) \in P_\ell(s,\Delta) \leftarrow \text{polytopic family of polynomials}$$

define a vector

$$p(\delta) = \begin{bmatrix} p_0(\delta) \\ p_1(\delta) \\ \vdots \\ p_n(\delta) \end{bmatrix} \in \mathbb{R}^{n+1}$$

↓ the corresponding polynomial

$$p(s,\delta) = p_0(\delta) + p_1(\delta)\, s + \cdots + p_n(\delta)\, s^n$$

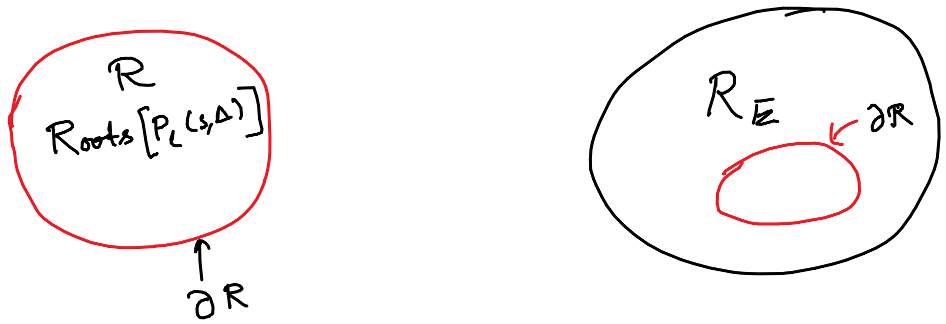
Let $P_\ell(\Delta)$ be the collection of all $p(\delta)$.

↪ $P_\ell(\Delta) := \{ p(\delta) \in \mathbb{R}^{n+1} \} \leftarrow$ polytope

- Assumption - * : The sign of $p_n(\delta)$ is constant over $P_\ell(\Delta) \subset \mathbb{R}^{n+1}$, either positive or negative always.

→ Edge Theorem

Let $P_\ell(\Delta) \subset \mathbb{R}^{n+1}$ be a polytope, associated with the polytopic family of polynomials $P_\ell(s,\Delta)$, and satisfies Assumption-∗ (that is, the degree of $p(s,\delta)$ remain same in $P_\ell(s,\Delta)$ for all $\delta \in \Delta$ or equivalently, sign of $p_n(\delta)$ is constant over $P_\ell(\Delta)$). Then, the boundary ∂R of the root set of $P_\ell(s,\Delta)$, i.e., $\text{Roots}[P_\ell(s,\Delta)]$ is contained in the root set of the exposed edges of $P_\ell(\Delta)$.



when $R := \text{Roots}[P_\ell(s,\Delta)]$

$R_E :=$ Roots of the polynomials which are in the exposed edges of $P_\ell(s,\Delta)$,

→ The Edge theorem says that the boundary of root set of the polytope of polynomials $P_\ell(s,\Delta)$ is contained in the root set of the polynomials, which are in the exposed edges of $P_\ell(s,\Delta)$, i.e. exposed edges of $P_\ell(\Delta)$.

→ Hence, according to the Edge theorem, the robust stability of polytope of polynomials $P_\ell(s,\Delta)$, can be determined by checking the stability of the family of polynomials, which are in the exposed edges of $P_\ell(s,\Delta)$.

→ Note that the exposed edges of $P_\ell(s,\Delta)$ (or equivalently $P_\ell(\Delta)$) are part of all pairwise convex combinations of the vertex polynomials of $P_\ell(s,\Delta)$, i.e. vertices of $P_\ell(\Delta)$.

$\downarrow$

Hence, for robust stability checking of polytopic family of polynomials, we check the stability of all pairwise convex combinations of the vertex polynomials of $P_\ell(s,\Delta)$.

→ Example :

Consider the previous example, which has 4 vertex polynomials : $p(s,\delta^{v_1})$, $p(s,\delta^{v_2})$, $p(s,\delta^{v_3})$, $p(s,\delta^{v_4})$

So $P_\ell(s,\Delta) = \text{conv}\{p(s,\delta^{v_1}), p(s,\delta^{v_2}), p(s,\delta^{v_3}), p(s,\delta^{4})\}$

The convex combinations of all vertex polynomials:

$$p_\lambda^1(s,\delta) = (1-\lambda)\,p(s,\delta^{v_1}) + \lambda\, p(s,\delta^{v_2}) \qquad \lambda \in [0,1]$$

$$p_\lambda^2(s,\delta) = (1-\lambda)\,p(s,\delta^{v_1}) + \lambda\, p(s,\delta^{v_3})$$

$$\vdots$$

continue this way

Observe that

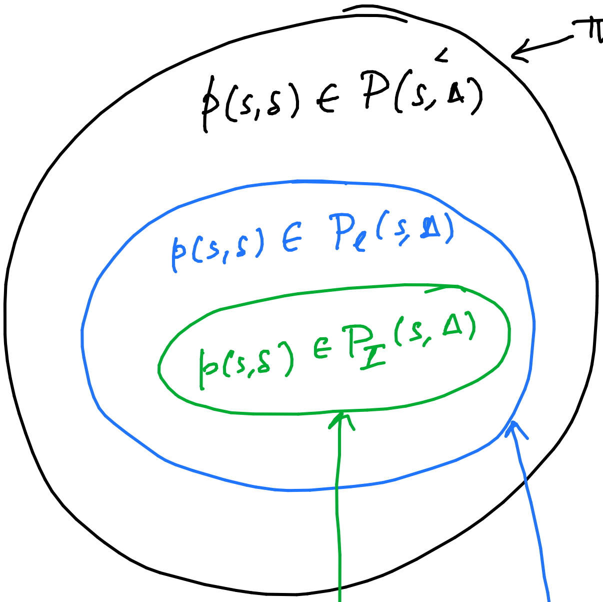
$$p_\lambda^1(s,\delta) = (1-\lambda)\underbrace{\left[p(s,\delta^{v_1}) + \frac{\lambda}{1-\lambda}\,p(s,\delta^{v_2})\right]}_{\bar{p}_\lambda^1(s,\delta)}$$

- $\text{Roots}\left[p_\lambda^1(s,\delta)\right] = \text{Roots}\left[\bar{p}_\lambda^1(s,\delta)\right]$

Hence, stability of $p_\lambda^1(s,\delta)$ is equivalent to the stability of $\bar{p}'_\lambda(s,\delta)$.

$$\rightarrow \quad \bar{p}_\lambda^1(s,\delta) = p(s,\delta^{v_1}) + \underline{\eta}\, p(s,\delta^{v_2}) \quad \text{where} \quad \eta \in [0,\infty)$$

Hence the stability of $p_\lambda^1(s,\delta)$ can be analyzed by using <u>Root-locus technique</u>.



The set of uncertain polynomials where the co-efficients of the polynomials are <u>continuous</u> functions of δ.

↳ <u>stability tests</u>

- Boundary Crossing theorem
- Zero Exclusion principle
- Using Hurwitz matrices

Polytope of polynomials,

↳ <u>stability test</u>

- Edge Theorem

↑ testing the stability of line segments using Root-locus technique

Interval Polynomials

↳ <u>stability test</u>

- Khanitonov's Result

↑ Testing the Stability of 4 corner polynomials