Lecture - 15

• System Norms :

We will consider two type of systems:

(i) Real rational strictly proper, stable functions

$G(s) \rightarrow \lim_{s \to \infty} (G(s)) = 0$

also the poles of $G(s)$ are in the open left half of complex plane.

$$G(s) = \frac{2}{s+1}$$

(ii) Real rational proper and stable functions

$\lim_{s \to \infty} G(s) = d$ where $d \in \mathbb{R}$

Ω^s_{spr}

(set of strictly proper, stable transfer funct")

$R\mathcal{H}_2$

Ω^s_{pr}

set of proper and stable transfer functions.

$R\mathcal{H}_\infty$

$\Omega^s_{spr} \subset \mathcal{H}_2$ $\qquad$ $\Omega^s_{pr} \subset \mathcal{H}_\infty$

$\downarrow$

$R\mathcal{H}_2$ $\qquad$ Hardy space $\qquad$ $R\mathcal{H}_\infty$

We will define norms ($\mathcal{H}_2$ & $\mathcal{H}_\infty$) in the spaces Ω^s_{spr} & Ω^s_{pr}.

- **$\mathcal{H}_2$ - norm**

This is defined in Ω^s_{spr} ($R\mathcal{H}_2$)

Let say $G(s)$ is an element in Ω^s_{spr}.

$$G(s) \xrightarrow{\text{Impulse response}} g(t)$$

$g(t) \in \mathcal{L}_2$ space $\quad \|g(t)\|_2 < \infty$

- $\mathcal{H}_2$ norm of $G(s)$ is the $\mathcal{L}_2$ norm of its impulse response, i.e. $\|g(t)\|_2$.

$G(s)$ is a transfer function

$$\|G(s)\|_{\mathcal{H}_2} := \left[\int_0^\infty (g(t))^2 dt\right]^{1/2}$$

Transform to frequency dom $\quad \downarrow$ Parseval's theorem

$$\|G(s)\|_{\mathcal{H}_2} := \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega\right]^{1/2}$$

- Let $G(s)$ be a transfer function matrix, The impulse response of $G(s)$ is

$$g(t) = \begin{bmatrix} g_{11}(t) & g_{12}(t) & \cdots & g_{1m}(t) \\ g_{21}(t) & g_{22}(t) & \cdots & g_{2m}(t) \\ \vdots & & & \vdots \\ g_{n1}(t) & g_{n2}(t) & \cdots & g_{nm}(t) \end{bmatrix}$$

$$\|G(s)\|_{\mathcal{H}_2} = \left[\int_0^\infty \sum_{i=1}^{n} \sum_{j=1}^{m} (g_{ij}(t))^2 \, dt \right]^{1/2}$$

$$= \left[\int_0^\infty Tr\left[g^T(t)\, g(t) \right] dt \right]^{1/2}$$

$Tr(A) = \sum_{i=1}^{n} a_{ii}$ ← Trace of matrix A

In frequency domain:

$$\|G(s)\|_{\mathcal{H}_2} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} Tr\left[\underset{\text{complex conjugate Transpose}}{G^*(j\omega)}\, G(j\omega) \right] d\omega \right]^{1/2}$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{i=1}^{r} \sigma_i^2 (G(j\omega))\, d\omega \right) \right]^{1/2}$$

$\boxed{Tr(A^T A) = \sum_i \sigma_i^2 (A)}$ $\sigma_i(A)$ is i^{th} singular value of A.

→ **Result**

Let $\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\}$ be a minimal state space

realization of $G(s) \in \Omega^s_{spr}$. Then

$$\|G(s)\|^2_{\mathcal{H}_2} = Tr(B^T Q B) = Tr(C P C^T)$$

where Q is Observability Grammian and

P is Controllability Grammian

Q & P can be computed by solving following Lyapunov equations :

(i) $QA + A^TQ + C^TC = 0$

(ii) $AP + PA^T + BB^T = 0$

proof

Transfer func? $G(s) = C(sI-A)^{-1}B$

$\downarrow$

impulse response $g(t) = Ce^{At}B$.

so $\|G(s)\|_{\mathcal{H}_2}^2 = \|g(t)\|_2^2 = \int_0^\infty Tr\,[g^T(t)\,g(t)]\,dt$

$$= Tr\left[\int_0^\infty g^T(t)\,g(t)\,dt\right]$$

$$= Tr\left[\int_0^\infty B^Te^{A^Tt}C^TCe^{At}B\,dt\right]$$

$$= Tr\left[B^TQB\right]$$

when $Q = \int_0^\infty e^{A^Tt}C^TC\,e^{At}dt$ ← Observability Gramian

Since $Tr(A) = Tr(A^T)$ we have

$$\|G(s)\|_{\mathcal{H}_2}^2 = Tr\left[\int_0^\infty g(t)\,g^T(t)\,dt\right]$$

$$= Tr\left[\int_0^\infty Ce^{At}BB^Te^{A^Tt}C^T\,dt\right]$$

$$= \mathrm{Tr}\left[C P C^T \right]$$

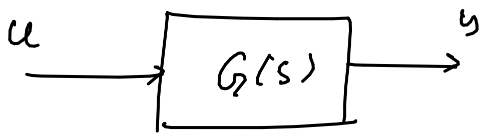
where $P = \int_0^\infty e^{At} B B^T e^{A^T t}$ ← Controllability Grammian

The observability Gramian & Controllability Grammian satisfy following Lyapunov eq:

$$\begin{cases} AP + PA^T + BB^T = 0 \\ A^T Q + QA + C^T C = 0 \end{cases}$$

→ Now consider the space $\Omega^s_{pr}(RH_\infty)$:

Let $G(s) \in \Omega^s_{pr}$



$u(t) \in L_2 \Rightarrow y(t) \in L_2$

We can define gain of the system as follows:

$$\|y\|_2^2 = \int_0^\infty (y(t))^2 dt$$

$$\|G\|_{gain} = \frac{\|y\|_2}{\|u\|_2} \quad \text{when } \|u(t)\|_2 \neq 0 .$$

- The greatest possible gain of $G(s)$ over the set of all $u(t) \in L_2$.

↳ is H_∞ norm of $G(s)$.

- $\mathcal{H}_\infty$-norm of $G(s)$ is the greatest possible gain over the set of $\mathcal{L}_2$-signals.

$G(s)$ is a t.f. matrix

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\substack{u(t)\in\mathcal{L}_2\\ \|u(t)\|_2\neq 0}} \frac{\|y\|_2}{\|u\|_2} = \sup_{\substack{u(t)\in\mathcal{L}_2\\ \|u(t)\|_2\neq 0}} \frac{\|Gu\|_2}{\|u\|_2}$$

$$\left[\ \|A\|_2 := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_{max}(A)\ \right]$$

$x \in \mathbb{R}^n$ — largest singular value of A

One can show that (By using Parseval's theorem and Contraction mapping theorem Khalil, non-linear syst)

$$\sup_{\|u(t)\|\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} = \sup_{\omega} \|G(j\omega)\|_2$$

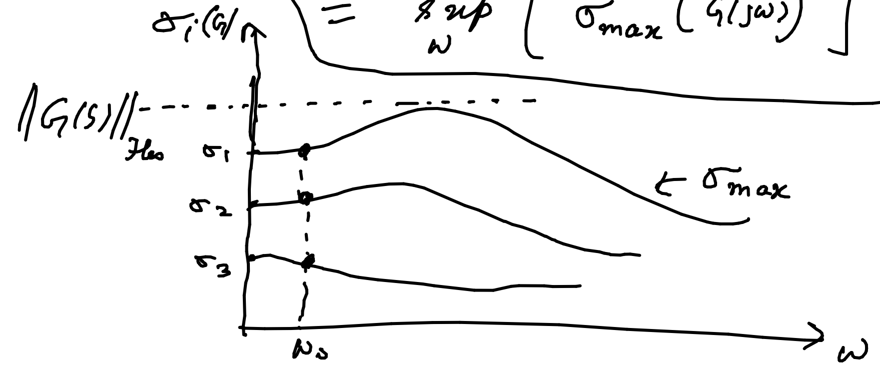
- one way is easy to show:

$$\sup_{\|u\|_2\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} \leq \sup_{\omega} \|G(j\omega)\|_2 \quad \text{(Parseval's Th)}$$

$$\|G(s)\|_{\mathcal{H}_\infty} = \sup_{\omega} \|G(j\omega)\|_2 = \sup_{\omega} \left[\lambda_{max}\left(G^*(j\omega)\,G(j\omega)\right)\right]^{1/2}$$

largest eigenvalue

$$= \sup_{\omega} \left[\sigma_{max}(G(j\omega))\right] \quad (0\leq\omega<\infty)$$



For a fixed ω_0 $G(j\omega_0)$ is a complex matrix & it has say 3 singular values.

- The maximum magnification is captured by the largest singular value of $G(j\omega)$. σ_{max}

- $\sigma_{max}(G(j\omega))$ is a function of ω.

Hence we consider the sup. of $\sigma_{max}(j\omega)$.

<u>For SISO</u> $(G(s))$

$$\|G(s)\|_{\mathcal{H}_\infty} = \sup_{\omega} |G(j\omega)| \leftarrow \text{magnitude}$$

↑ The maximum peak of Bode plot of $G(j\omega)$.

→ To approximate $\|G(s)\|_{\mathcal{H}_\infty}$, one can choose a set of frequencies $\{\omega_1, \omega_2 \cdots \omega_N\}$ and search for

fine grid of frequency points

$$\max_{1 \leq k \leq N} \left[\sigma_{max}\left(G(j\omega_k)\right)\right] \approx \|G(s)\|_{\mathcal{H}_\infty}$$

Hence for SISO case, $\|G(s)\|_{\mathcal{H}_\infty} \approx \max_{\omega} |G(j\omega)|$, and one can obtain this by using Bode plot.

- Computing Bound on $\mathcal{H}_\infty$-norm :

→ Result : Let $\gamma > 0$ and a stable linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

& the associated t.f.m $G(s) = C(sI - A)^{-1}B + D$. Then,

$\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ if and only if $\sigma_{max}(D) < \gamma$ and the following matrix

$$H = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix} \quad \text{---} (*)$$

where $R = \gamma^2 I - D^TD$,

has no eigenvalues on the imaginary axis.

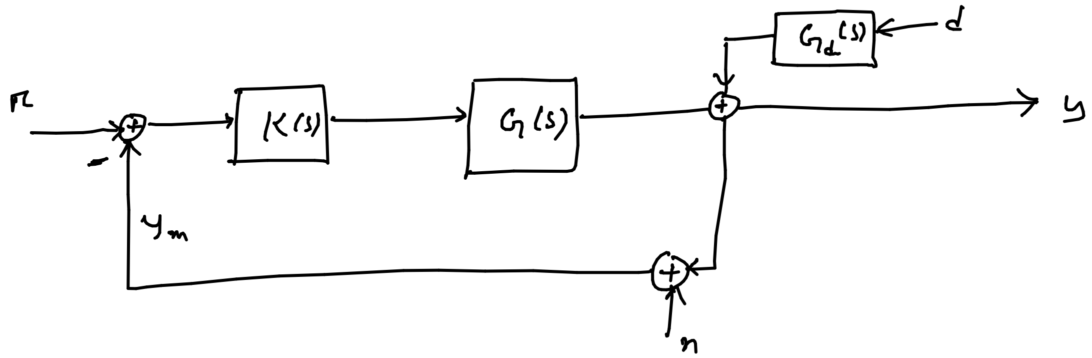
the result become simpler when $D = 0$

For $D = 0$,

$$H = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$ ← it should not have eigenvalues on imaginary axis.

↳ An algorithm to compute $\mathcal{H}_\infty$ norm based on the above result is available in [Zhou, Doyle and Glover].

→ Sensitivity and Complementary Sensitivity



$$u = K(r - y - n)$$

$$y = GKr - GKy - GKn + G_d d$$

$$\Rightarrow (I + GK)y = GKr - GKn + G_d d$$

$$\Rightarrow \quad y = \underbrace{(I+GK)^{-1}GK}_{T} r - \underbrace{(I+GK)^{-1}GK}_{T} n + \underbrace{(I+GK)^{-1}}_{S} G_d d$$

T: Complementary Sensitivity function

S: sensitivity function

- Loop transfer funct: $L = GK$
- Sensitivity funct $S = (I + L)^{-1}$
- Complementary Sensitivity fn $T = (I + L)^{-1} L$

$$\boxed{S + T = I}$$

on SISO case

$$\boxed{S + T = 1}$$

- Some commments on Sensitivity & Complementary Sensitivity functions:
(SISO case)

(i) when $r = n = 0$, & $G_d = 1$ $\Big\} \rightarrow y = S d$

- Hence, disturbance d gets attenuated at frequency ω_0, when $|S(j\omega_0)|$ is small.

- $|S(j\omega)|$ is small when $|L(j\omega)|$ is large.

$\downarrow$

Hence for <u>disturbance attenuation</u>, it is necessary to sharpe the loop gain $|L(j\omega)|$ such that it is large over those frequencies (where disturbance attenuation is required).

(ii) when $\left.\begin{matrix} r = 0 \\ d = 0 \end{matrix}\right\} \rightarrow y = -Tn$ where $T = \dfrac{L}{1+L}$

- Hence to reduce the effect of noise 'n' on output for some frequency range, $|T(j\omega)|$ must be kept small over those frequency ranges.

But $|S(j\omega) + T(j\omega)| = 1$ for $\forall \omega$

→ Note that the disturbances are usually of low frequency signals and noises are of high frequency signals.

↓

- Since $S(j\omega) + T(j\omega) = 1 \;\; \forall\, \omega$, it is a good idea to keep the $|S(j\omega)|$ to small at low frequency range. and $|T(j\omega)|$ to small at high frequency range.

↓ gives rise to concept of

<u>Loop Shaping</u>

→ <u>Loop Shaping</u> :

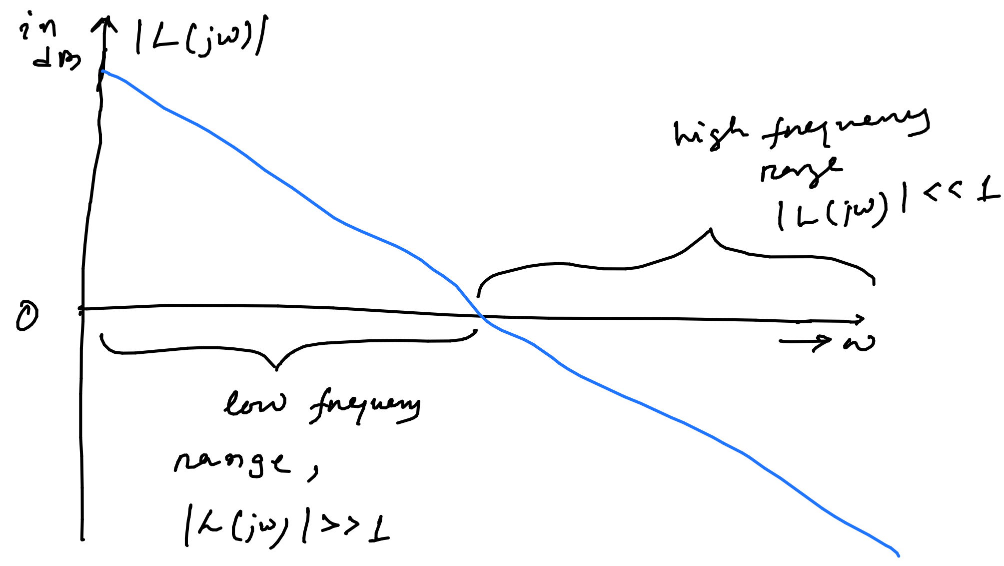
The problem of determining a feedback control $K(s)$ such that the loop gain frequency response function $|L(j\omega)|$ has a suitable shape over the desired frequency ranges.

- At low frequencies : In this frequency range we design S to be small and T to be close to 1.

Since $S = \frac{1}{1+L}$, the objective of keeping small S can be achived by making $|L(j\omega)| >> 1$ at low frequency range.

- At high frequency range: At this frequency range we design K to keep T small and S close to 1. This can be achived by making the loop gain $|L(j\omega)| << 1$ at high frequency region.

→ A typical loop gain plot:



Bode plot of $L(j\omega)$

→ Bode Sensitivity Integral

Suppose that loop gain $L(s)$ has at least two more poles than zeros. Suppose also that $L(s)$ has N_p - RHP poles at locations p_i. Then, for closed loop stability the sensitivity function must satisfy:

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = \pi \left[\underset{\uparrow}{Re(p_1)} + Re(p_2) + \cdots + Re(p_{N_p}) \right]$$

Real part of p_i

If $L(s)$ has no pole in RHP, then for closed loop stability

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = 0$$

(The area of sensitivity reduction must be equal to the sensitivity increase)

↓

- Sensitivity reduction in one frequency range will be reflected as sensitivity increase in some other frequency range.

↑

(water-bed property)