Lecture-16

Unstructured Uncertainty:

$\Omega_{pr}^{s} \rightarrow$ proper & stable T-fs

Uncertainty

Parametric/ structured

- $\delta \in \mathbb{R}$ is an uncertain parameter
- Ex: $P(\delta, s) = \dfrac{1}{s^2 + 2\delta_1 s + 3\delta_2}$

$\delta_1 \in \mathbb{R}$
$\delta_2 \in \mathbb{R}$

Non-parametric/ Unstructured

- $\Delta(s) \in \Omega_{pr}^{s}$ is an uncertain parameter

Additive

$G_\Delta(s) = G_0(s) + \Delta(s)$

Multiplicative

$G_\Delta(s) = \left(1 + \Delta(s)\right) G_0(s)$

$G_0(s)$ is the nominal plant.

For stability study & control design we need to quantify the uncertainty in some sense

- Quantificat^n: $\delta_i \in [\delta_i^-, \delta_i^+]$

- $\Delta(s) \in \Omega_{pr}^{s}$

$\|\Delta(s)\|_{\mathcal{H}_\infty} \leq \gamma$

where $\gamma > 0$.

$$\mathcal{P}_a := \left\{ G_\Delta(s) := G_0 + \Delta(s) \;\middle|\; \Delta(s) \in \mathcal{R}^s_{pR} \;\&\; \|\Delta(s)\|_{\mathcal{H}_\infty} < r \right\}$$

$\downarrow$

$\Delta(s)$ — Weighting function ($W_1(s)$ & $W_2(s)$ both belongs to $\mathcal{R}^s_{pR}$)

$\downarrow$

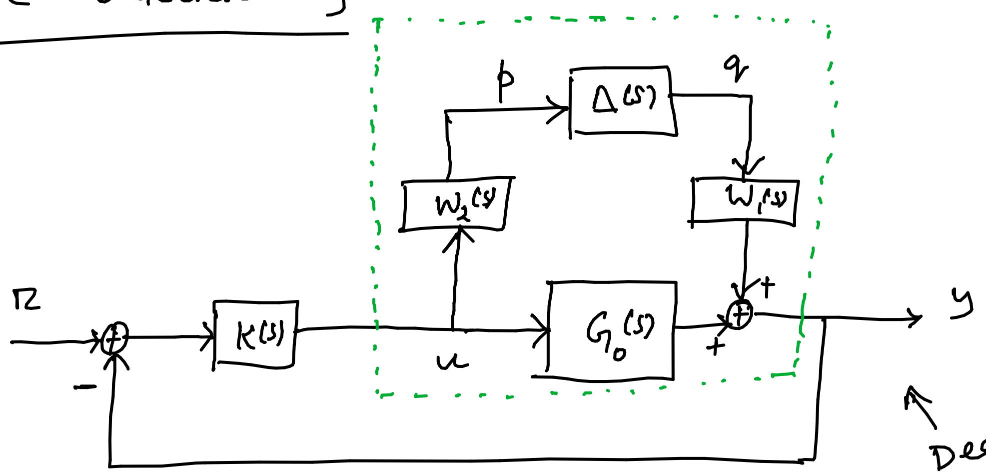
$W_1(s)\,\Delta(s)\,W_2(s)$

$$\mathcal{P}_a := \left\{ G_\Delta(s) = G_0(s) + W_1(s)\Delta(s)W_2(s) \;\middle|\; \|\Delta(s)\|_{\mathcal{H}_\infty} \le r \right\}$$

$$\mathcal{P}_m := \left\{ G_\Delta(s) = \big(I + W_1(s)\Delta(s)W_2(s)\big)\, G_0(s) \;\middle|\; \|\Delta(s)\|_{\mathcal{H}_\infty} \le r \right\}$$

multiplication uncertain plant family

- Additive Uncertainty



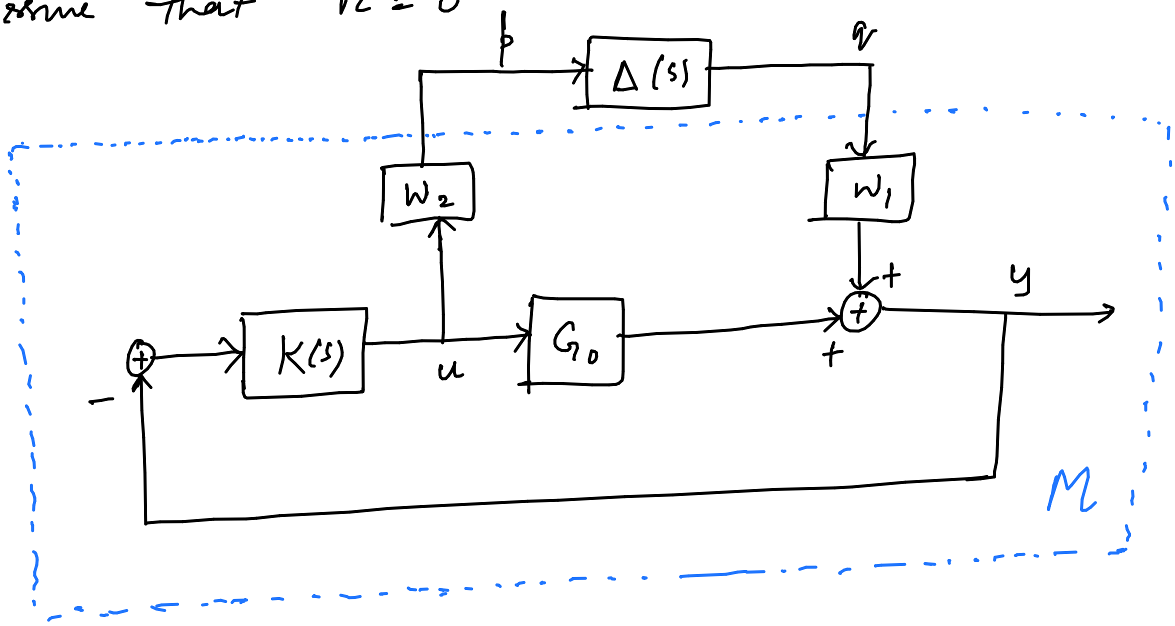
Deals with Additive uncertainty

$$y = G_0 u + W_1 q$$

$$= G_0 u + W_1 \Delta W_2 u = \underbrace{(G_0 + W_1 \Delta W_2)}_{G_\Delta(s)} u$$

$$y = G_\Delta(s)\, u$$

Assume that $r = 0$



For the block M, the o/p → p

& i/p → q

we bring a relationship between p & q

$$y = G_0 u + W_1 q \qquad\qquad p = W_2 u$$

$$= -G_0 K y + W_1 q \qquad\qquad = -W_2 K y$$

$$\Rightarrow (I + G_0 K) y = W_1 q \qquad\qquad = -W_2 K (I + G_0 K)^{-1} W_1 q$$

$$\Rightarrow y = (I + G_0 K)^{-1} W_1 q$$

$$p = -W_2 K (I + G_0 K)^{-1} W_1 q$$

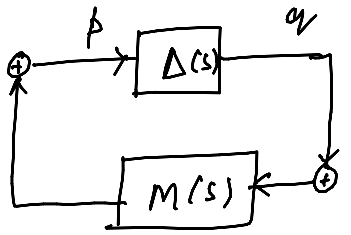
$$= -W_2 K S_0 W_1 q \; ;$$ where S_0 is

$S_0 = (I + G_0 K)^{-1}$

sensitivity fun^n

So $\boxed{M = -W_2 K S_0 W_1}$

The additive feedback interconnection can be represented as

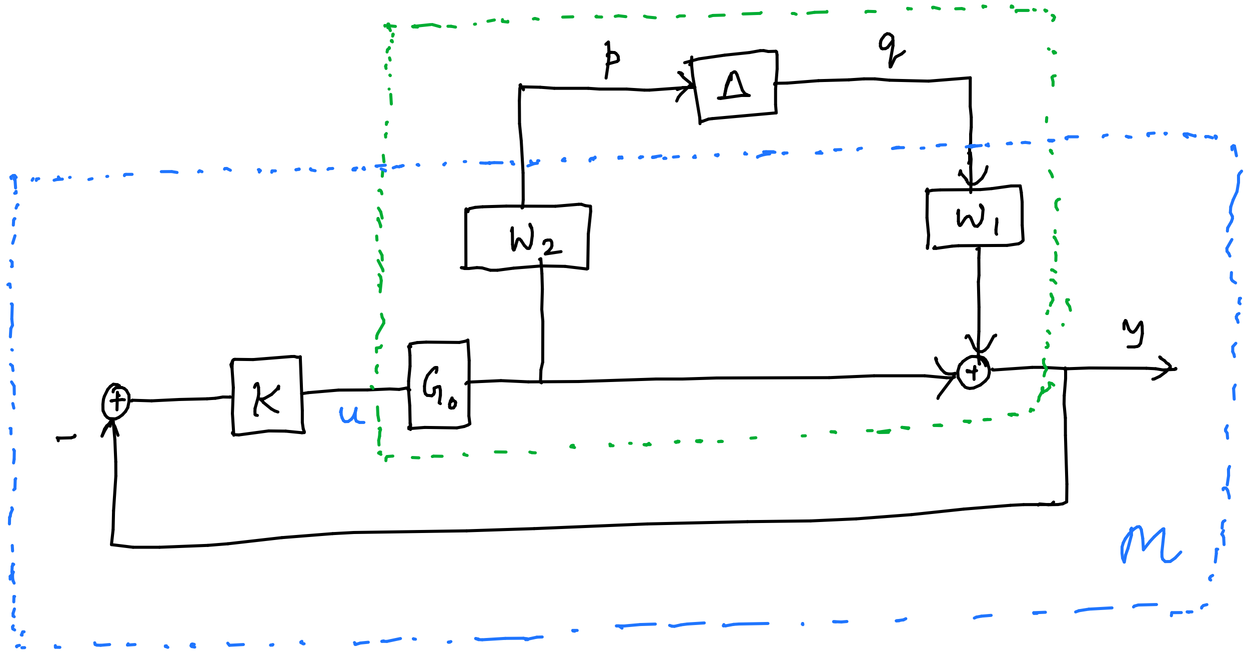


$\rightarrow$ ($M-\Delta$) structure

when $M = -W_2 K S_0 W_1$

- Multiplicative Uncertainty:

for uncertainty representation



$$y = G_0 u + W_1 q$$

$$= G_0 u + W_1 \Delta W_2 G_0 u = \underbrace{(I + W_1 \Delta W_2) G_0}_{G_\Delta(s)} u$$

Multiplicative uncertainty

$\downarrow$ Construct the $M-\Delta$ structure

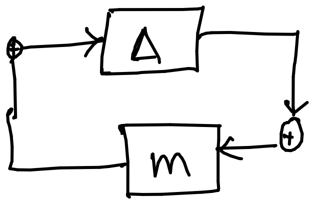
$$y = G_0 u + W_1 q \qquad\qquad p = W_2 G_0 u$$

$$\Rightarrow y = (I + G_0 K)^{-1} W_1 q \qquad = -W_2 G_0 K y$$

$$= -W_2 G_0 K (I + G_0 K)^{-1} W_1 q$$

$$\boxed{p = -W_2 T_0 W_1 q} \rightarrow \boxed{M = -W_2 T_0 W_1}$$

where $T_0 = G_0 K (I + G_0 K)^{-1} \leftarrow$ Complementary sensitivity fun^n



→ Stability study of $M - \Delta$ structure:

↳ we use Small gain theorem for the robust stability analysis

• Small Gain Theorem

Let us assume that $M(s) \in \Omega_{pr}^{s}$.

Then the feedback interconnection is well-posed and internally stable for all $\Delta(s) \in \Omega_{pr}^{s} (R\mathcal{H}_\infty)$ with $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{\gamma}$ if and only if $\|M(s)\|_{\mathcal{H}_\infty} \leq \gamma$.

(proof → see Zhou, Doyle book)

Based on Small Gain Theorem:

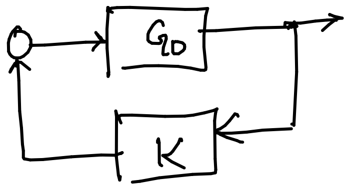
→ Result: Let $K(s)$ be a stabilizing controller for the nominal plant $G_0(s)$. Let

$$G_\Delta(s) = G_0(s) + W_1(s)\Delta(s)W_2(s) \quad \text{(additive uncertainty)}$$

where $\Delta(s) \in \Omega^s_{pr}$ with $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{\gamma_a}$.

Then the feedback interconnection ($M-\Delta$ str) is robustly stable for all $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{\gamma_a}$ if and only if $\|W_2 K S_0 W_1\|_{\mathcal{H}_\infty} \leq \gamma_a$.

Proof



(For SISO case)

Since K is a stabilizing controller for G_0, we can say that

$$S_0 = \frac{1}{1+G_0 K} \quad \text{is stable \& proper}$$

(Recall that for internal stability all of the 4 t.f. are stable.) So

$$K S_0 = \frac{K}{1+G_0 K} \quad \text{is also stable \& proper}$$

$W_2 K S_0 W_1$ ← is also stable & proper

Since W_2 & W_1 are stable & proper

$\|W_2 K S_0 W_1\|_{\mathcal{H}_\infty}$ can now be defined. Thus the result directly follows from SGT.

The design parameter is $K(s)$.

→ Result :

Let $K(s)$ be a stabilizing controller for the nominal plant $G_0(s)$. Let

$$G_\Delta(s) = (I + W_1 \Delta W_2) G_0 \quad \text{(Multiplicative uncertainty.}$$

where $\Delta(s) \in \Omega^s_{pr}$. Then the feedback interconnection is well-posed and internally stable (Robustly stable) for all $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{r}$ if and only if $\|W_2 T_0 W_1\|_{\mathcal{H}_\infty} \leq r$.

For Robust stability

- Additive uncertainty → $\|W_2 K S_0 W_1\|_{\mathcal{H}_\infty} \leq r$
- Multiplicative uncertainty → $\|W_2 T_0 W_1\|_{\mathcal{H}_\infty} \leq r$.

provided $\|\Delta\|_{\mathcal{H}_\infty} < \frac{1}{r}$

↓

→ A common platform to study all kind of uncertainties :

↓

Linear Fractional Transformation

→ Linear Fractional Transformation:

Let us consider a complex matrix M which is partitioned as follows:

$$M = \left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{21} & M_{22} \end{array}\right]$$

Let Δ_l & Δ_u be other two complex matrices.

Then define

- Lower LFT:

$$\boxed{F_l(M, \Delta_l) := M_{11} + M_{12}\Delta_l (I - M_{22}\Delta_l)^{-1} M_{21}}$$

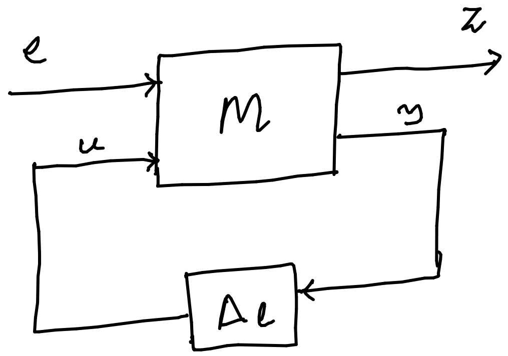
provided $(I - M_{22}\Delta_l)^{-1}$ exists.

- Upper LFT

$$\boxed{F_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u (I - M_{11}\Delta_u)^{-1} M_{12}}$$

provider $(I + M_{11}\Delta_u)^{-1}$ exists.

- The LFTs are called "well-defined" iff $(I - M_{11}\Delta_u)^{-1}$ & $(I - M_{22}\Delta_l)^{-1}$ exist.



$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}$$

Bring a relationship bet z & e

$$u = \Delta_l y$$

$$y = M_{21} e + M_{22} u$$

$$= M_{21} e + M_{22} \Delta_l y$$

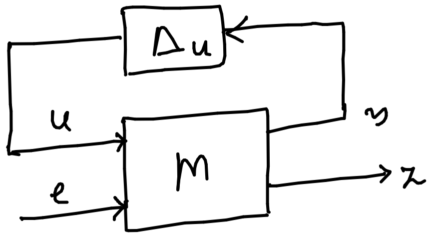
$$\Rightarrow y = (I - M_{22}\Delta_l)^{-1} M_{21} e$$

$$z = M_{11} e + M_{12} u$$

$$= M_{11} e + M_{12} \Delta_l y$$

$$z = M_{11} e + M_{12}\Delta_l (I - M_{22}\Delta_l)^{-1} M_{21} e$$

$$= \underbrace{\left(M_{11} + M_{12}\Delta_l (I - M_{22}\Delta_l)^{-1} M_{21}\right)}_{F_l(M, \Delta_l)} e$$



$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix}, \quad u = \Delta_u y$$

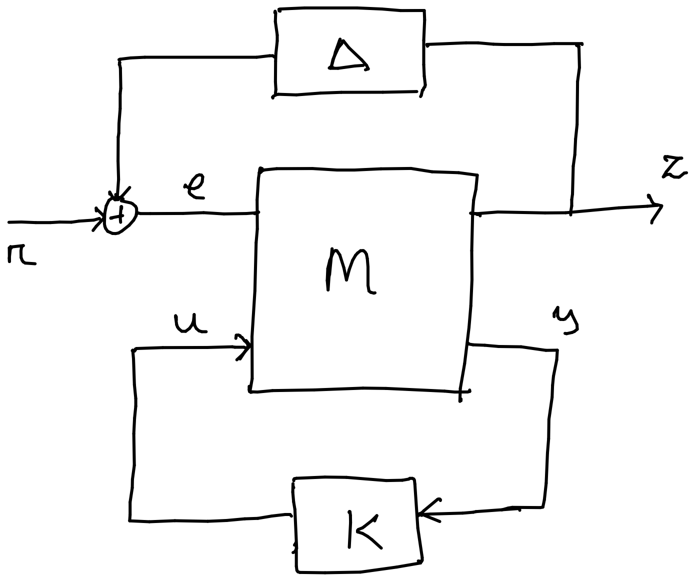
$$y = M_{11} u + M_{12} e$$

$$\Rightarrow y = (I - M_{11}\Delta_u)^{-1} M_{12} e$$

$$\begin{aligned} z &= M_{21} u + M_{22} e \\ &= M_{21} \Delta_u y + M_{22} e \\ &= \underbrace{\left(M_{22} + M_{21}\Delta_u (I - M_{11}\Delta_u)^{-1} M_{12}\right)}_{F_u(M, \Delta_u)} e \end{aligned}$$

$$z = F_u(M, \Delta_u) e$$

→ A common configuration for Robust stability :-



$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Assume $r = 0$, & $K = 0$

when $K=0$, we are dealing with upper LFT

So $\quad z = M_{11} e + M_{12} u$

$$= M_{11} \Delta z + M_{12} u$$

$$\Rightarrow z = (I - M_{11}\Delta)^{-1} M_{12} u$$

$$y = M_{21} e + M_{22} u$$

$$= M_{21} \Delta z + M_{22} u$$

$$= M_{21} \Delta (I - M_{11}\Delta)^{-1} M_{12} u + M_{22} u$$

$$= \underbrace{\left[M_{22} + M_{21}\Delta (I - M_{11}\Delta)^{-1} M_{12}\right]}_{F_u(M,\Delta)} u$$

$y \rightarrow u$ relation was for representing the type of uncertainty we deal.

$$y = G_\Delta(s)\, u$$

when $\boxed{G_\Delta(s) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}}$

Let us now chose $M_{11} = 0 \quad M_{12} = W_2 \quad M_{21} = W_1$

& $M_{22} = G_0$

$$M = \begin{bmatrix} 0 & W_2 \\ W_1 & G_0 \end{bmatrix} \rightarrow G_\Delta(s) = G_0 + W_1 \Delta W_2$$

↑ additive uncertainty

Let us use

$$M = \left[\begin{array}{c|c} 0 & W_2 G_0 \\ \hline W_1 & G_0 \end{array}\right]$$

Then $G_\Delta(s) = G_0 + W_1 \Delta W_2 G_0$

$$= (\underline{I} + W_1 \Delta W_2)\, G_0$$

Multiplicative uncertainty