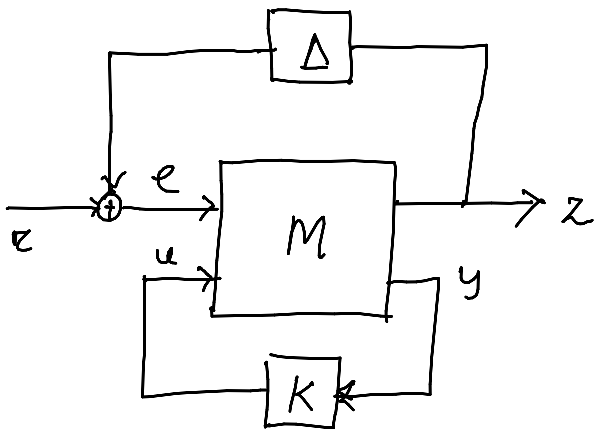
Lecture - 17



$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}$$

We assume that $\Delta = 0$, we obtain the mapping fn for the rest of the block:

$$y = M_{21} e + M_{22} u$$

$$= M_{21} r + M_{22} K y \quad (\text{since } \Delta = 0,\ e = r)$$

$$\Rightarrow y = (I - M_{22} K)^{-1} M_{21} r$$

$$z = M_{11} e + M_{12} u$$

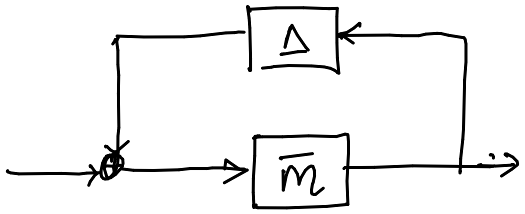
$$= M_{11} r + M_{12} K y$$

$$= M_{11} r + M_{12} K (I - M_{22} K)^{-1} M_{21} r$$

$$= \underbrace{\left[M_{11} + M_{12} K (I - M_{22} K)^{-1} M_{21} \right]}_{\text{Lower LFT}} r$$

$$z = \bar{M} r = \bar{M} e$$

where $\bar{M} = M_{11} + M_{12} K (I - M_{22} K)^{-1} M_{21}$



→ $(M-\Delta)$ structure of feedback interconnection.

- Recall that for additive uncertainty:

$$M = \left[\begin{array}{c|c} 0 & W_2 \\ \hline W_1 & G_0 \end{array}\right]$$

So $\overline{M} = W_2 K (I - G_0 K)^{-1} W_1$

$= W_2 K S_0 W_1$

For robust stability:

Assuming that $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{r}$

we need $\boxed{\|W_2 K S_0 W_1\|_{\mathcal{H}_\infty} \leq r}$.

- For multiplicative uncertainty,

$$M = \left[\begin{array}{c|c} 0 & W_2 G_0 \\ \hline W_1 & G_0 \end{array}\right]$$

$\overline{M} = W_2 G_0 K (I - G_0 K)^{-1} W_1$

$= W_2 T_0 W_1$

For robust stability we need $\boxed{\|W_2 T_0 W_1\|_{\mathcal{H}_\infty} \leq r}$.

for $\|\Delta(s)\|_{\mathcal{H}_\infty} < \frac{1}{r}$

→ Robustness Analysis for SISO case using Nyquist plot

- Additive Uncertainty

$$G_\Delta(s) = G_0(s) + W_a(s)\Delta(s)$$

where $G_0 \rightarrow$ nominal plant
$W_a \in \mathcal{R}^s_{pr}$
(weighting function)

Assume that $|\Delta(j\omega)| \leq 1$ for all $\omega \in \mathbb{R}$.

$$|G_\Delta(j\omega) - G_0(j\omega)| = |W_a(j\omega)\,\Delta(j\omega)| \quad \text{for all } \omega \in \mathbb{R}$$

$$= |W_a(j\omega)|\,|\Delta(j\omega)|$$

$$\leq |W_a(j\omega)| \quad \text{———} \; (*)$$

$$\underline{\|\Delta(j\omega)\|_{H_\infty} \leq 1} \equiv \underline{|\Delta(j\omega)| \leq 1 \text{ for } \forall \omega \in \mathbb{R}}$$

How do you define a ball in the complex plane

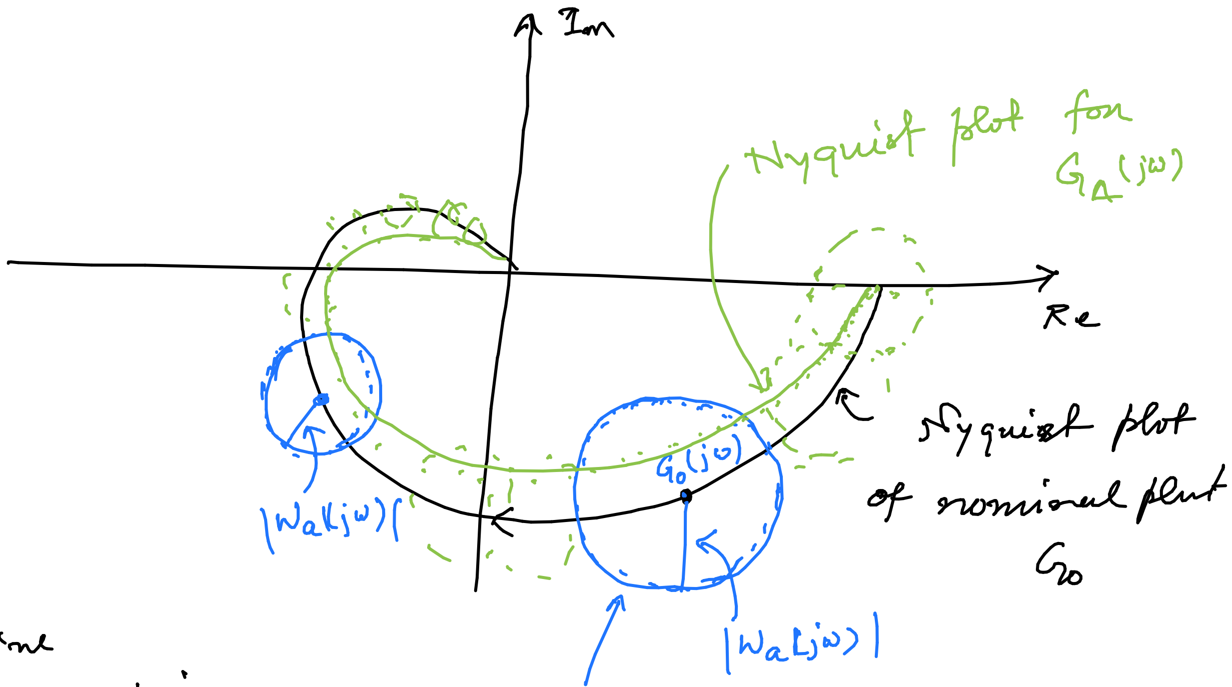
disk

$|z - z_0| \leq \epsilon_r$ with center z_0 & radius ϵ_r

For ⊛ ,

$$|G_\Delta(j\omega) - G_0(j\omega)| \leq |W_a(j\omega)| \quad \forall\, \omega \in \mathbb{R}$$

Disc in complex plane having
center $G_0(j\omega)$
& radius $|W_a(j\omega)|$



Since $|W_a(j\omega)|$ is dependent on ω
the radius of the ball also dependent on ω

→ Multiplicative Uncertainties

Let the nominal plant be $G_0(s)$.

And assume that $K(s)$ is a stabilizing controller for $G_0(s)$.

The nominal loop gain $L_0 = G_0 K$, assume that it is stable.

For multiplicative uncertainty

$$G_\Delta(s) = \left(1 + W_m(s)\Delta(s)\right) G_0$$

$$G_\Delta(s) K = \left(1 + W_m(s)\Delta(s)\right) G_0 K$$

$$\downarrow$$

$$\boxed{L_\Delta(s) = \left(1 + W_m(s)\Delta(s)\right) L_0(s)}$$

$$\Rightarrow \quad L_\Delta(s) - L_0(s) = W_m(s)\Delta(s) L_0(s)$$

$$\downarrow$$

$$|L_\Delta(j\omega) - L_0(j\omega)| = |W_m(j\omega)\Delta(j\omega) L_0(j\omega)| \quad \forall \omega \in \mathbb{R}$$

$$= |W_m(j\omega) L_0(j\omega)| \, |\Delta(j\omega)|$$

Assume that $|\Delta(j\omega)| \leq 1 \quad \forall \omega \in \mathbb{R}$

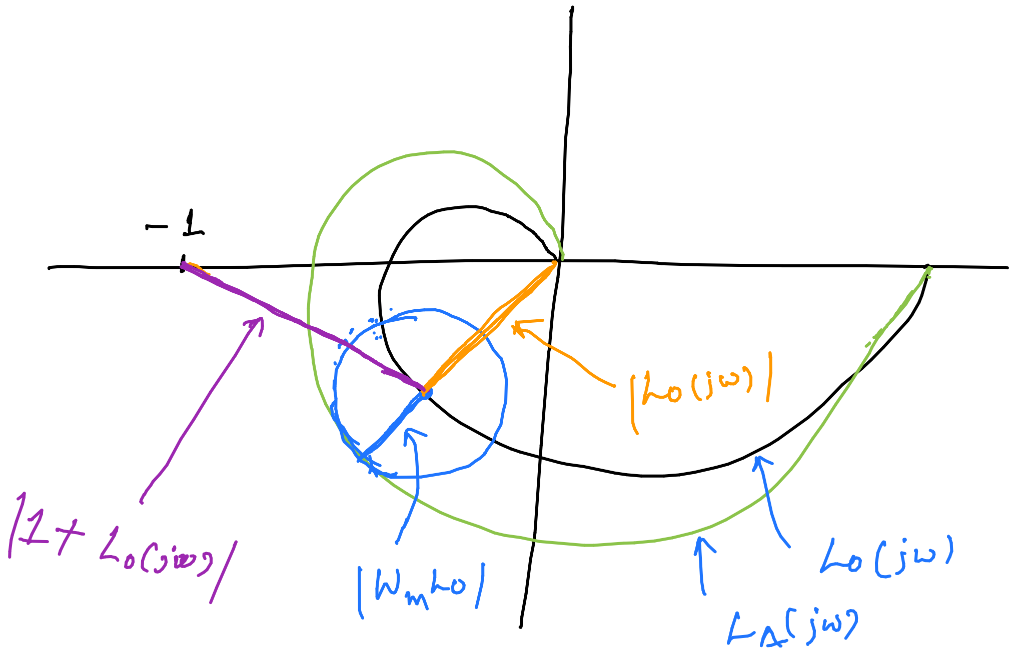
$$\rightarrow |L_\Delta(j\omega) - L_0(j\omega)| \leq |W_m(j\omega) L_0(j\omega)| \quad \forall \omega \in \mathbb{R}$$

(**)

For $(**)$, $|L_\Delta - L_0| \leq |W_m L_0|$ $\forall\, \omega \in \mathbb{R}$

$\downarrow$

the disc having center at L_0 & radius $|W_m L_0|$



For <u>robust stability</u> $L_\Delta(j\omega)$ should not encircle the point -1.

$\Updownarrow$

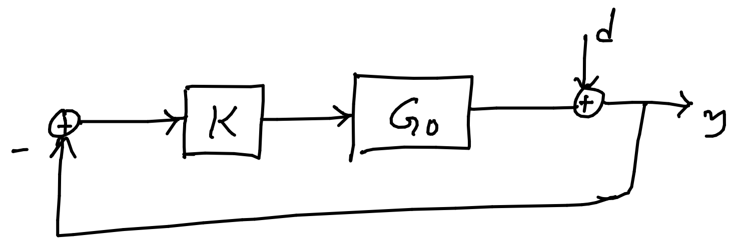
$$|W_m L_0| < |1 + L_0| \quad \text{for all } \omega \in \mathbb{R}$$

$$\Leftrightarrow \left| \frac{W_m L_0}{1 + L_0} \right| < 1 \quad \forall\, \omega \in \mathbb{R}$$

$$\Leftrightarrow |W_m T_0| < 1 \quad \forall\, \omega \in \mathbb{R}$$

For multiplicative uncertainty robust stability cond^n $\nearrow$ $|W_m(j\omega)\, T_0(j\omega)| < 1$ for $\forall\, \omega \in \mathbb{R}$.

- Distrubance Rejection:



$y = S_o d$ where S_o is sensitivity function

Assum that the disturban is scaled s.t

$|d(j\omega)| \leq 1 \quad \forall \omega \in \mathbb{R}$

For disturbance rejection, we need

$|S_o(j\omega)| < 1$

Instead of taking sensitivity funct^n, it is preferable to take weighted sensitivity funct^n. $\longrightarrow W_s(s)\, S_o(s)$

For disturbance reject^n we need

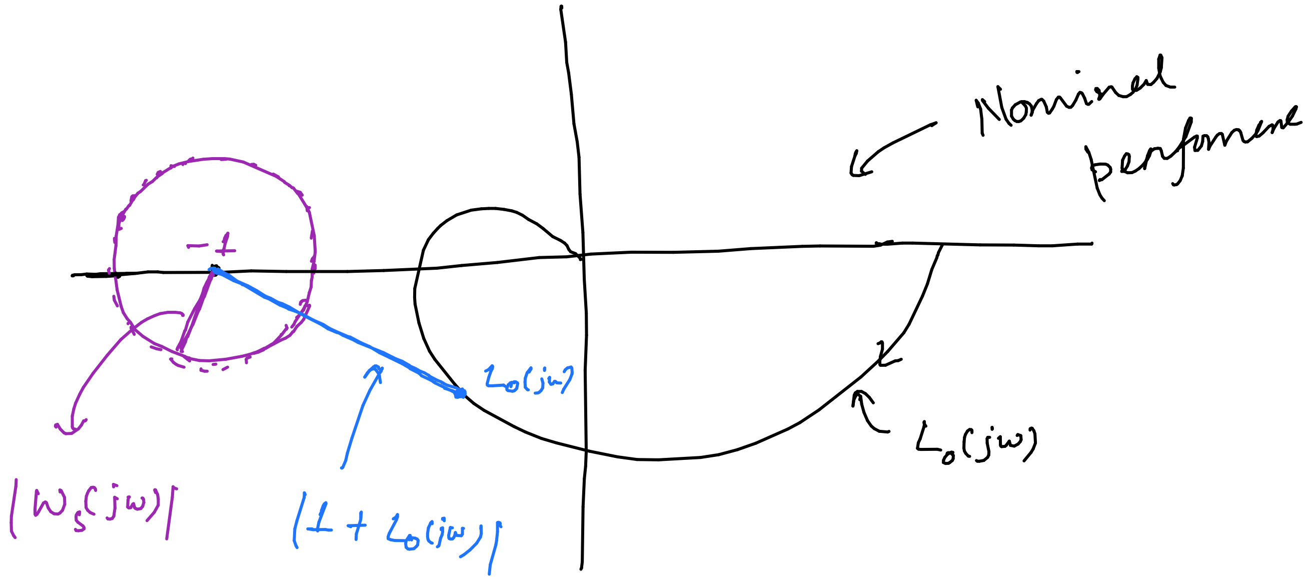
$$|W_s(j\omega)\, S_o(j\omega)| < 1 \quad \forall \omega \in \mathbb{R}$$

$$\Leftrightarrow \quad |W_s(j\omega)| < |1 + L_o(j\omega)| \quad \forall \omega \in \mathbb{R}$$

since $S_o = \dfrac{1}{1+L_o}$

For disturbance rejection

$$|W_s(j\omega)| < |1 + L_o(j\omega)| \quad \forall \omega \in \mathbb{R}$$



Here for disturbance rejection we need

$$|W_s(j\omega)| < |1 + L_o(j\omega)|$$

the Nyquist plot of $L_o(j\omega)$ should not enter into the disc having center at -1 & radius $|W_s(j\omega)|$.

for achieving nominal performance.

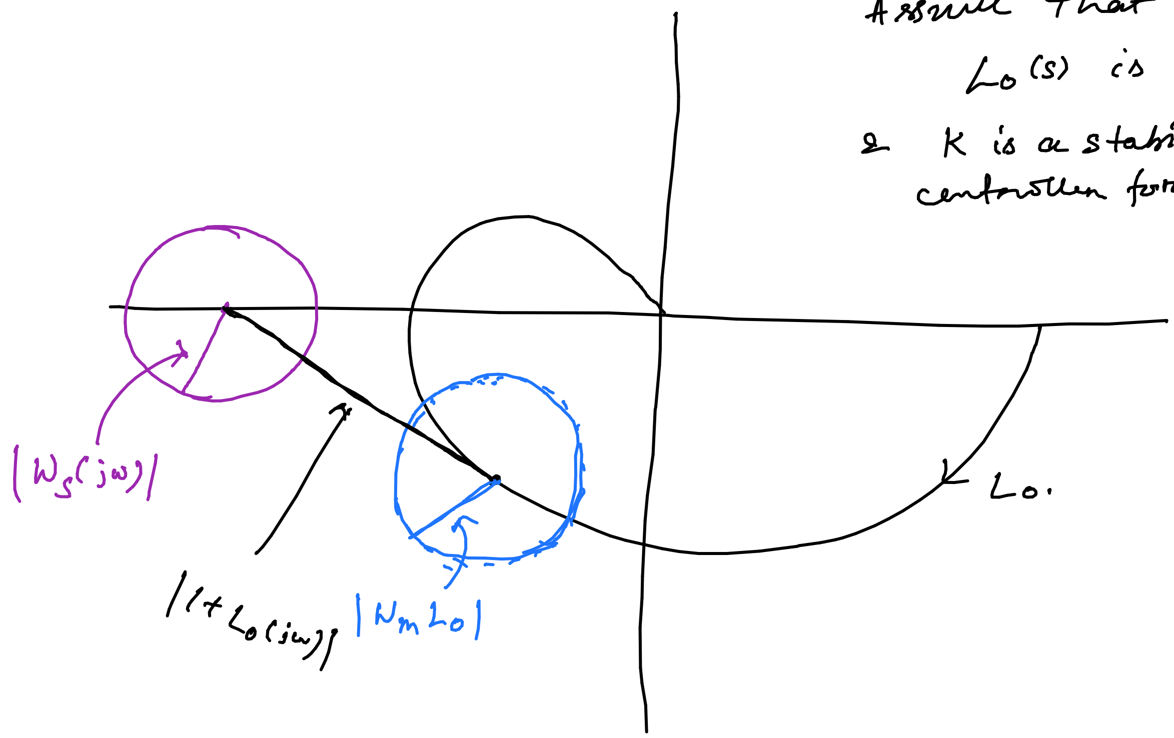
For Robust stability & Disturbance Rejection

Robust Performance

Assume that

$L_o(s)$ is stable.

& K is a stabilizing controller for G_o.



For robust performance, the two balls must not intersect.

This can be achieved by ensuring that

$$|W_s(j\omega)| + |W_m(j\omega)L_o(j\omega)| < |1 + L_o(j\omega)| \quad \forall \omega \in \mathbb{R}$$

$$\Leftrightarrow \left|\frac{W_s}{1+L_o}\right| + \left|\frac{W_m L_o}{1+L_o}\right| < 1 \quad \forall \omega \in \mathbb{R}$$

$$\boxed{\Leftrightarrow \quad |W_s S_o| + |W_m T_o| < 1 \quad \forall \omega \in \mathbb{R}}$$

for robust performance.